Intro to Algorithms

Asymptotic Notation

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Algorithms

- Correctness
- Running Time

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Today's Lecture

- Issues with Computing Runtime

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- Asymptotic Runtimes: Advantages and Disadvantages

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- Issues with Computing Runtime
- Asymptotic Runtimes: Advantages and Disadvantages
- Big-*O* Expressions

Fibonacci numbers

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$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

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more than 200 digits long!

Exponential Growth

Lemma

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Proof

By Induction on n.

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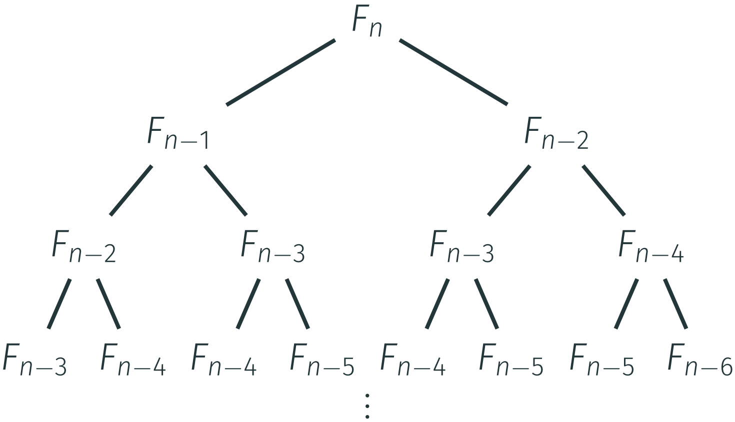

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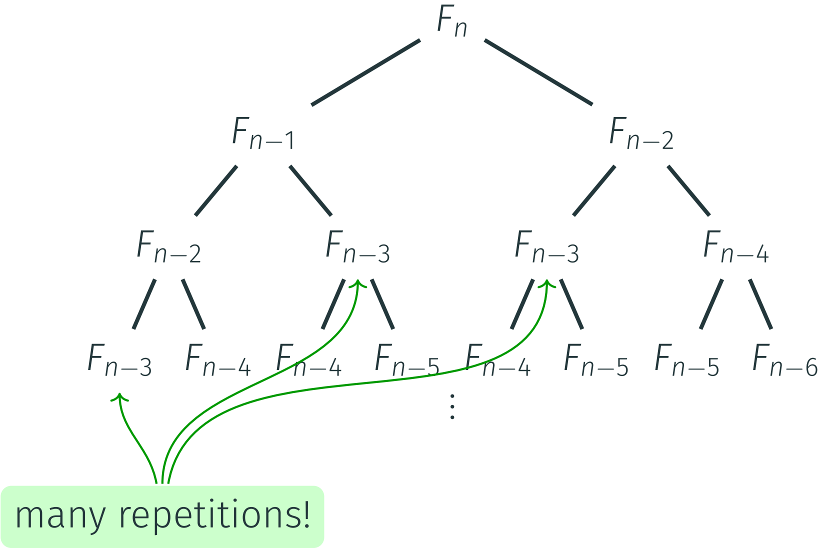

Running time

Let $T(n)$ denote the number of computer steps taken by `FibRec`(n). Then

$$T(n) = 2 \text{ for } n \leq 1$$

and

$$T(n) = T(n-1) + T(n-2) + 3 \text{ for } n > 1.$$

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Hence $T(n) \geq F_n \geq 2^{n/2}$.

$$T(100) > 2.86 \cdot 10^{21}$$

Takes more than 90,000 years to compute

FASTER ALGORITHM

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function Fib(n)
create an array F[0 . . . n]
F[0] ← 0, F[1] ← 1
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$T(n) = 2n + 2$ $\quad$ $T(100) = 202$

Visualization

http://www.cs.usfca.edu/~galles/visualization/DPFib.html

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- Improved algorithm is so much faster
- We want algorithms that are efficient for large inputs
- The right algorithm makes all the difference

Running Time

Closer look

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Exact Running Time

- Processor's speed
- Compiler
- System Architecture
- Memory Hierarchy

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- Measure runtime ignoring constants

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- All of that modifies runtime by a constant factor
- Measure runtime ignoring constants
- How runtime scales with n: $\sqrt{n}, n, n^2, 1.5^n, \ldots$

Comparing various running times and input sizes

	n	$n \log n$	n^2	2^n

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$n = 10^6$	1 sec	1 sec	17 min	
$n = 10^9$	1 sec	30 sec	30 yr	
max n for 1 sec	10^9	10^7	$10^{4.5}$	30

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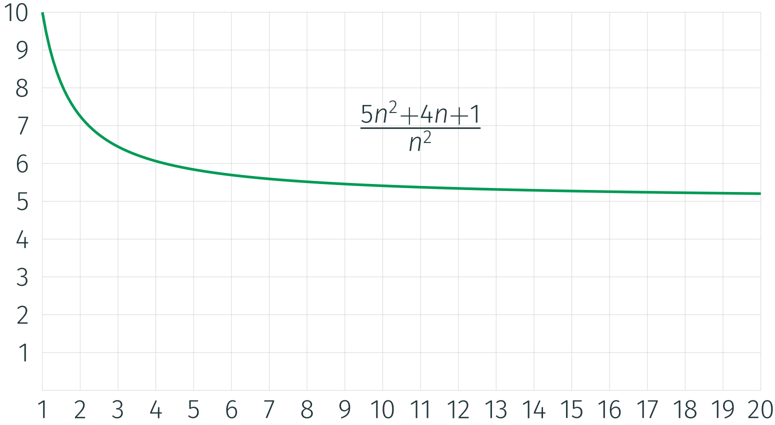
- Clears up notation for running time: n^3 instead of $7n^2 - 10n + \frac{n^3}{5} + \sin(x)$
- Loses (important) information about constant multiples
- For large n, difference between n and $100n$ is insignificant comparing to difference between n and n^3

Asymptotic Notation

Growth rate

$5n^2 + 4n + 1$ has the same growth rate as n^2



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- $5n^2 + 4n + 1 \le 5n^2 + 4n^2 + n^2 = 10n^2 = O(n^2)$
- $5n^2 + 4n + 1 = O(n^2)$,
  $100n^2 - 5 = O(n^2)$,
  $n^2/50 - 70n = O(n^2)$

Quiz

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- $2^{n+1} = O(2^n)$?
- $n \log_2 n = O(n)$?

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$O(n)$ operations

$O(1)$ operations

Total: $O(n) + O(1) + O(n) \cdot O(n) + O(1) = O(n^2)$ operations

Missing Number

- Compute sum of all elements in stream:

$$s = x_1 + \ldots x_n$$

- Sum of all numbers in range $\{0, \ldots, n\}$ is $S = \frac{n(n+1)}{2}$
- Missing number is $S - s = \frac{n(n+1)}{2} - s$

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- Running time is $O(n)$

Majority Element

- $\mathsf{count} \leftarrow 0; \quad \mathsf{m} \leftarrow \perp$
- For each element x_i of Stream:
  - If $\mathsf{count} = 0$, then $\mathsf{m} \leftarrow x_i$ and $\mathsf{count} \leftarrow 1$
  - ElseIf $x_i = \mathsf{m}$, then count ++
  - Else count --
- Return m

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  - Else $\mathsf{count}{-}{-}$
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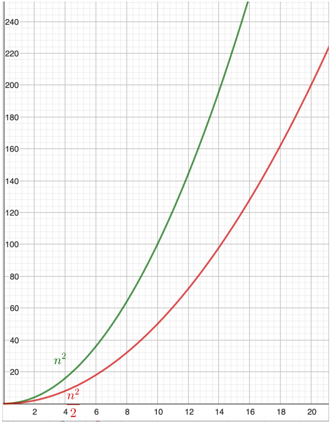

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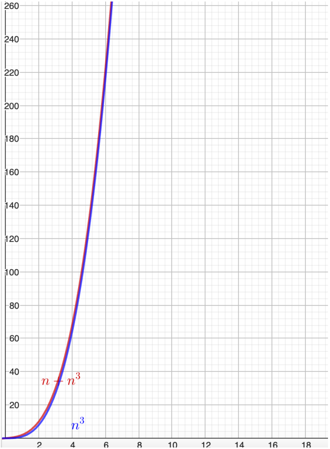

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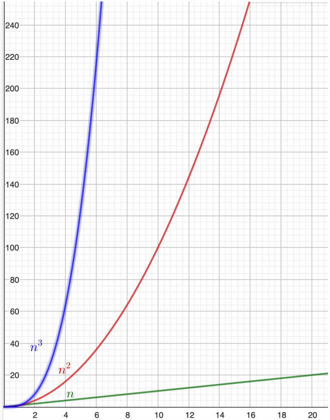

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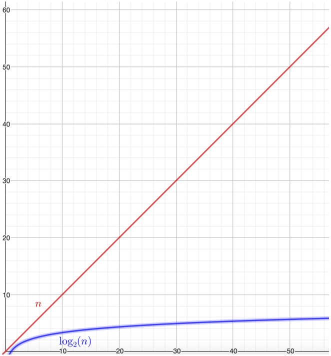

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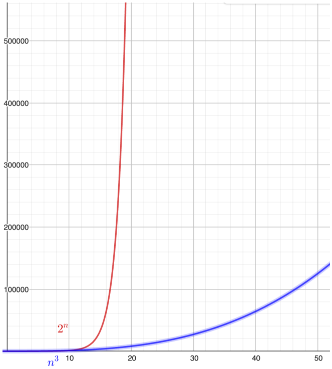

Quiz

- Sort the following functions by their orders of growth
  - n^2
  - $\log n$
  - 2^n
  - $n \log n$
  - n
  - $\sqrt{n}$

Frequently used functions

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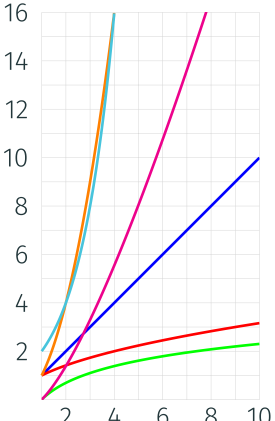

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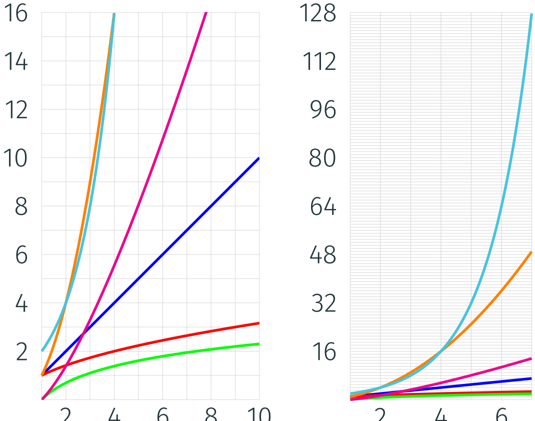

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- $f = \Omega(g)$ (f grows no slower than g, $f \succeq g$), if $g = O(f)$
- $f = \Omega(g)$ if there are constants $c, n_0 > 0$ such that $f(n) \geq c \cdot g(n)$ for all $n \geq n_0$
- $5n^2 + 3n = \Omega(n)$, $\frac{n^3}{5} - 10n - 5 = \Omega(n^3)$

Asymptotically Tight Bounds

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- $17n^4 - 3n + 10 = \Theta(n^4)$

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- $n^3 = \omega(n)$, $n = \omega(\log n)$

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- $\Theta(n^c)$ for a constant $c > 0$ Polynomial functions
- $\Theta(c^n)$ for a constant $c > 1$ Exponential functions