Gems of TCS

Easy and Hard Problems

Sasha Golovnev

August 23, 2023

This Course

- Theoretical/Mathematical viewpoint

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- Theoretical/Mathematical viewpoint
- Topic overview

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    - Algorithms

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    - Learning

Theoretical Computer Science

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$P \implies Q$

Mathematical logic

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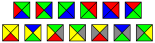
Mathematical
logic



Computability
theory

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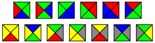

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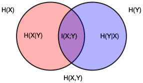

Information theory

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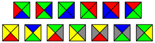

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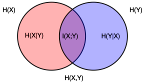

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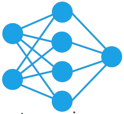

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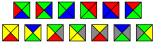

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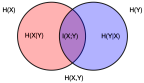

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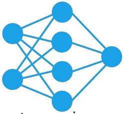

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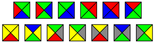

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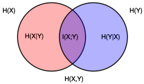

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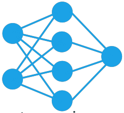

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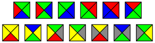

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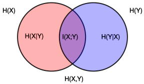

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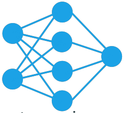

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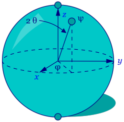

Quantum Algorithms

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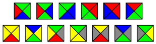

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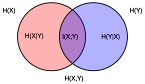

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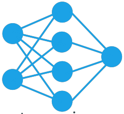

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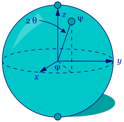

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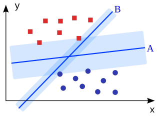

Machine learning

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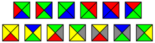

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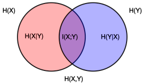

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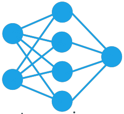

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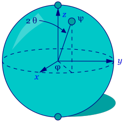

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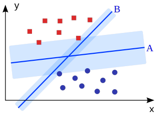

Machine learning

Data Science

Administrative Info

- Classes: MW 11:00am–12:15pm, Healy 103

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- email: `alexgolovnev+gems@gmail.com`

Course Begins

- Running time of an algorithm

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- Complexity class **P**: Problems whose solution can be found efficiently

- Complexity class **NP**: Problems whose solution can be verified efficiently

The main open problem in Computer Science

Is P equal to NP?

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Millenium Prize Problem

Clay Mathematics Institute: $1M prize for solving the problem

- If P=NP, then all NP-problems can be solved in polynomial time.

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- If P≠NP, then there exist NP-problems that cannot be solved in polynomial time.

NP-complete Problems

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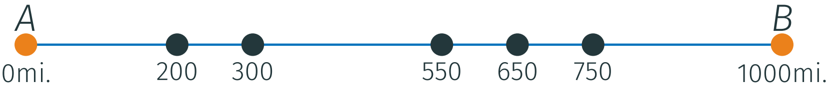
- If one NP-complete problem cannot be solved in polynomial time, then all NP-complete problems cannot be solved in polynomial time
- Later we'll show NP-complete problems exist!

Car Fueling

Car Fueling

Distance with full tank 300 mi.

Minimize the number of stops at gas stations



Break http://bit.ly/car-fueling

Example

Distance with full tank 300 mi.

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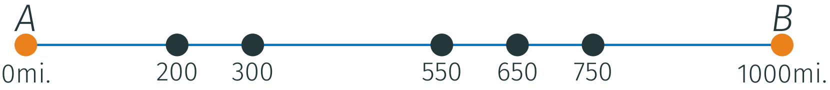

Car Fueling. Solution

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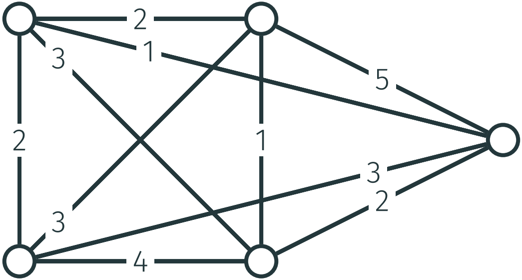
- Runs in linear time $O(n)$, where n is the size of the input (# of gas stations)
- Easy problem

Traveling Salesman Problem (TSP)

Traveling Salesman Problem

Given a complete weighted graph, find a cycle (or a path) of minimum total weight (length) visiting each node exactly once



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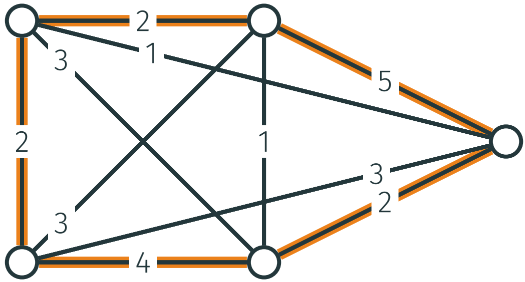

length: 15

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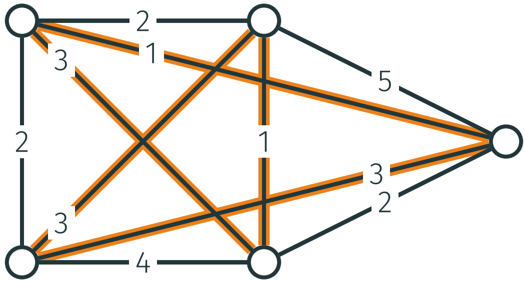

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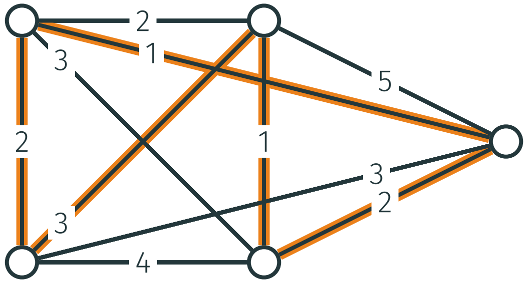

length: 9

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- Classical optimization problem with countless number of real life applications (we'll see soon)

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- No polynomial time algorithms known
- The best known algorithm runs in time 2^n

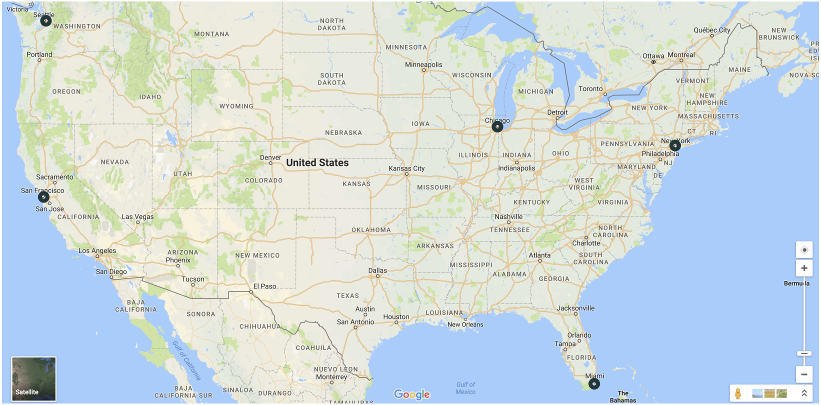
Delivering Goods

Need to visit several points. What is the optimal order of visiting them?

TRAVELING

Victoria
Seattle
WASHINGTON
MONTANA
NORTH DAKOTA
MINNESOTA
Québec City
NEW BRUNSWICK
Portland
Minneapolis
Ottawa
Montreal
MAINE
WISCONSIN
SOUTH DAKOTA
OREGON
IDAHO
MICHIGAN
Toronto
VERMONT
NOVA SCO
WYOMING
NEW HAMPSHIRE
NEW YORK
Detroit
Chicago
MASSACHUSETTS
IOWA
NEBRASKA
CT
RI
PENNSYLVANIA
New York
ILLINOIS
INDIANA
OHIO
Philadelphia
NEVADA
Denver
United States
NJ
Kansas City
Indianapolis
MARYLAND
UTAH
Sacramento
DE
COLORADO
KANSAS
WEST VIRGINIA
MISSOURI
San Francisco
KENTUCKY
VIRGINIA
San Jose
Las Vegas
CALIFORNIA
Nashville
TENNESSEE
OKLAHOMA
NORTH CAROLINA
Charlotte
ARKANSAS
Los Angeles
ARIZONA
NEW MEXICO
Atlanta
SOUTH CAROLINA
Phoenix
MISSISSIPPI
Dallas
San Diego
ALABAMA
GEORGIA
Tucson
El Paso
Bermuda
BAJA CALIFORNIA
TEXAS
Austin
LOUISIANA
Jacksonville
SONORA
Houston
San Antonio
New Orleans
CHIHUAHUA
Orlando
Tampa
Gulf of California
COAHUILA
FLORIDA
NUEVO LEON
Monterrey
Miami
Gulf of Mexico
BAJA CALIFORNIA SUR
SINALOA
DURANGO
TAMAULIPAS
The Bahamas
Satellite
Google

TRAVELING



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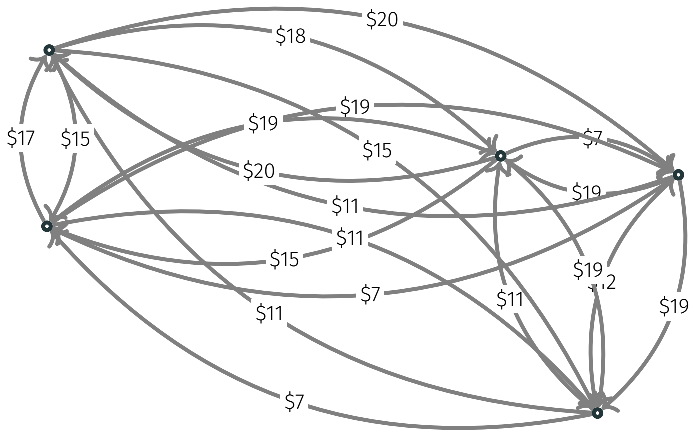
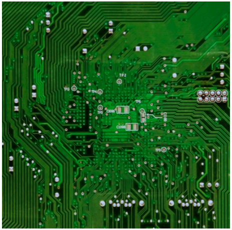
Traveling

Traveling

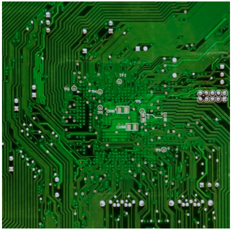


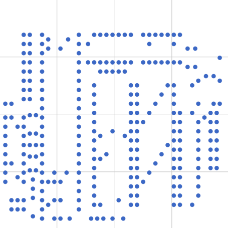
Drilling a Circuit Board



https://developers.google.com/optimization/routing/tsp/tsp

Drilling a Circuit Board





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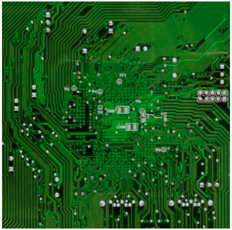

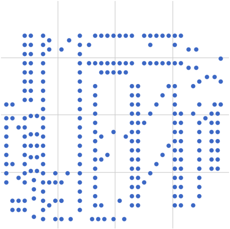

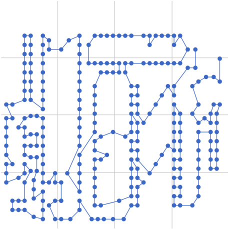

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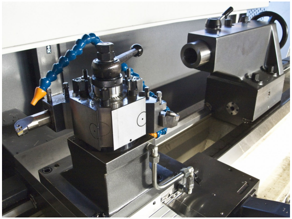
Processing Components

There are n mechanical components to be processed on a complex machine. After processing the i-th component, it takes t_{ij} units of time to reconfigure the machine so that it is able to process the j-th component. What is the minimum processing cost?



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- Euclidean TSP: instead of a complete graph, the input consists of n points $p_1 = (x_1, y_1), \ldots, p_n = (x_n, y_n)$ in the plane

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- Weights are symmetric: $d(p_i, p_j) = d(p_j, p_i)$
- Weights satisfy the triangle inequality: $d(p_i, p_j) \leq d(p_i, p_k) + d(p_k, p_j)$

Brute Force Search

- Finding the best permutation is easy: simply iterate through all of them and select the best one

Brute Force Search

- Finding the best permutation is easy: simply iterate through all of them and select the best one
- But the number of permutations of n objects is $n!$

$n!$: Growth Rate

n	$n!$
5	120
8	40320
10	3628800
13	6227020800
20	2432902008176640000
30	265252859812191058636308480000000

Satisfiability Problem (SAT)

SAT

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3) \wedge (x_2 \vee \neg x_3)$$

SAT

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3) \wedge (x_2 \vee \neg x_3)$$

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3) \wedge (x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3)$$

Applications of SAT

- Software Engineering
- Chip testing
- Circuit design
- Automatic theorem provers
- Image analysis
- ...

k-SAT

$$\begin{aligned}\phi(x_1, \ldots, x_n) = &(x_1 \vee \neg x_2 \vee \ldots \vee x_k) \quad \wedge \\ &\qquad\qquad \ldots \qquad\qquad\qquad \wedge \\ &(x_2 \vee \neg x_3 \vee \ldots \vee x_8)\end{aligned}$$

k-SAT

$$\begin{aligned} \phi(x_1, \dots, x_n) = & (x_1 \vee \neg x_2 \vee \dots \vee x_k) \quad \wedge \\ & \qquad \dots \qquad\qquad \wedge \\ & (x_2 \vee \neg x_3 \vee \dots \vee x_8) \end{aligned}$$

ϕ is satisfiable if

$$\exists x \in \{0,1\}^n \colon \phi(x) = 1 \,.$$

Otherwise, ϕ is unsatisfiable

k-SAT

$$\phi(x_1, \ldots, x_n) = (x_1 \vee \neg x_2 \vee \ldots \vee x_k) \wedge \\ \ldots \wedge \\ (x_2 \vee \neg x_3 \vee \ldots \vee x_8)$$

ϕ is satisfiable if

$$\exists x \in \{0,1\}^n \colon \phi(x) = 1 \,.$$

Otherwise, ϕ is unsatisfiable

k-SAT is SAT where clause length $\leq k$

k-SAT. Examples

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3) \wedge (x_2 \vee \neg x_3)$$

k-SAT. Examples

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3) \wedge (x_2 \vee \neg x_3)$$

$$(x_1) \wedge (\neg x_2) \wedge (x_3) \wedge (\neg x_1)$$

Queen of NP-complete problems

- Cook-Levin Theorem [Coo71, Lev73]: SAT can model non-deterministic Turing machine:

  SAT is NP-complete

Queen of NP-complete problems

- Cook-Levin Theorem [Coo71, Lev73]: SAT can model non-deterministic Turing machine: SAT is NP-complete

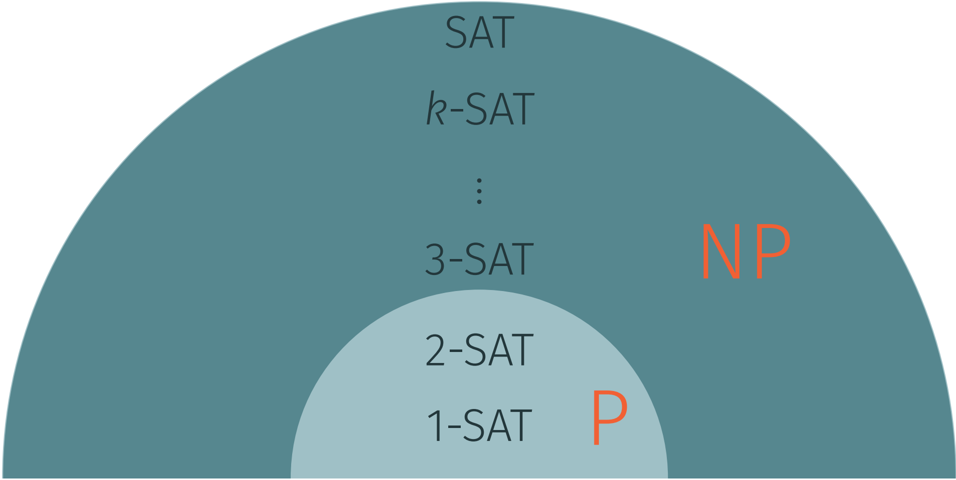
- 3-SAT is NP-complete

Queen of NP-complete problems

- Cook-Levin Theorem [Coo71, Lev73]: SAT can model non-deterministic Turing machine: SAT is NP-complete
- 3-SAT is NP-complete
- 2-SAT is in P

Complexity of SAT

Complexity of SAT



The SAT game

http://bit.ly/sat-game