# Gems of TCS

## Linear Programming

Sasha Golovnev

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- Optimization problems: among all solutions satisfying certain constraints find optimal one
- Find shortest cycle through all vertices
- Find optimal coloring
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- Linear programming: class of optimization problems where constraints and optimization criterion are linear functions

# Avoiding Scurvy

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- Orange has 50gm of vitamin C,
  grapefruit has 75gm of vitamin C,
  maximize daily vitamin C intake.

# Avoiding Scurvy. Plot

$$\max 2x + 3y$$

$$x + y \leq 2$$

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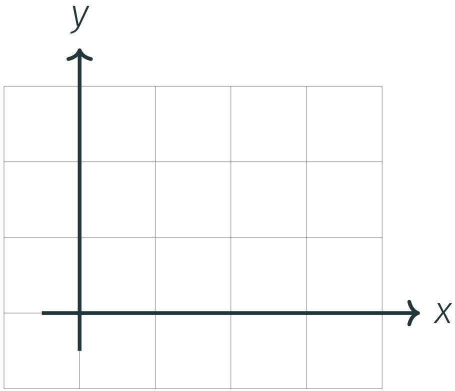

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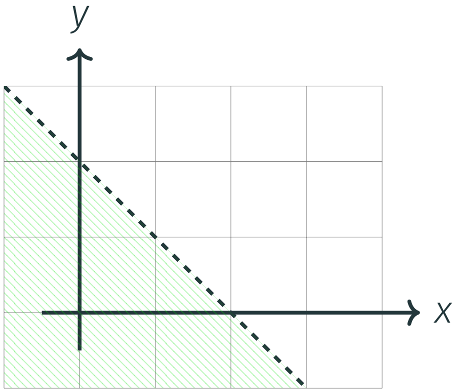

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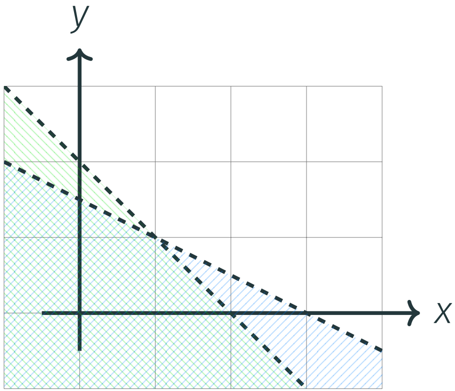

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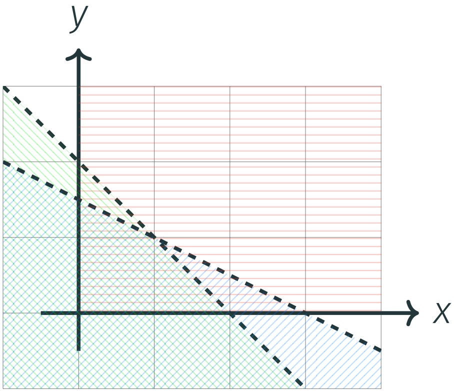

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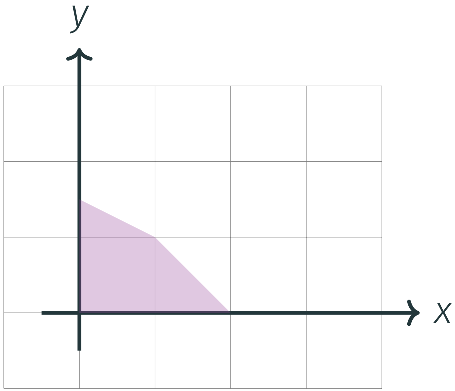

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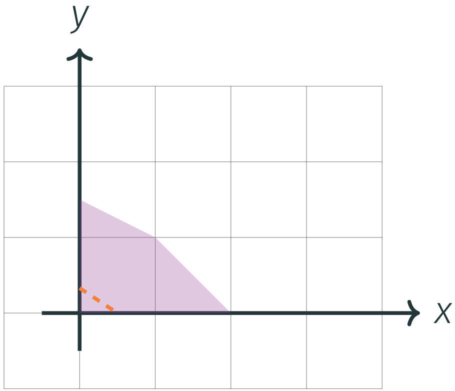

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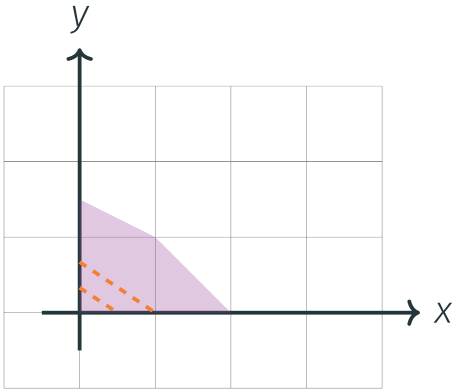

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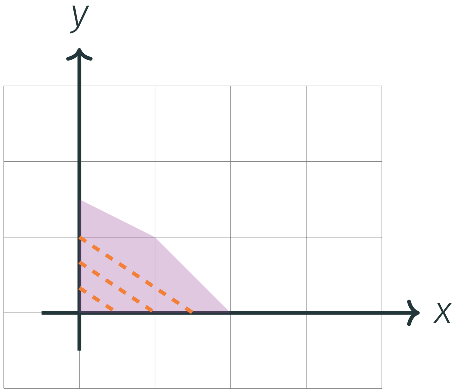

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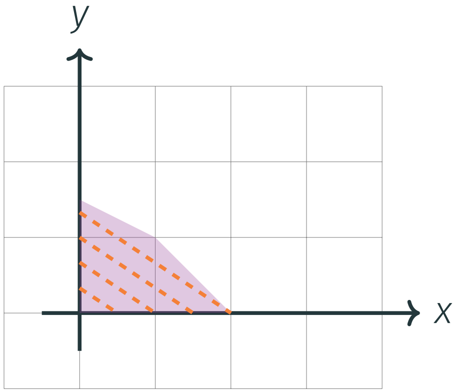

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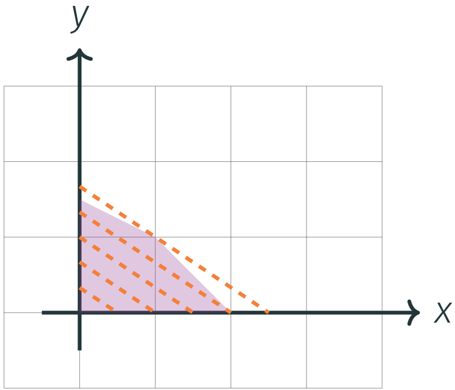

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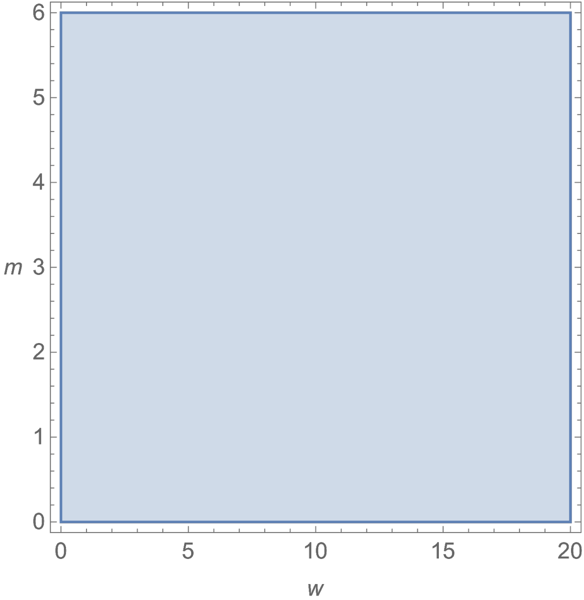
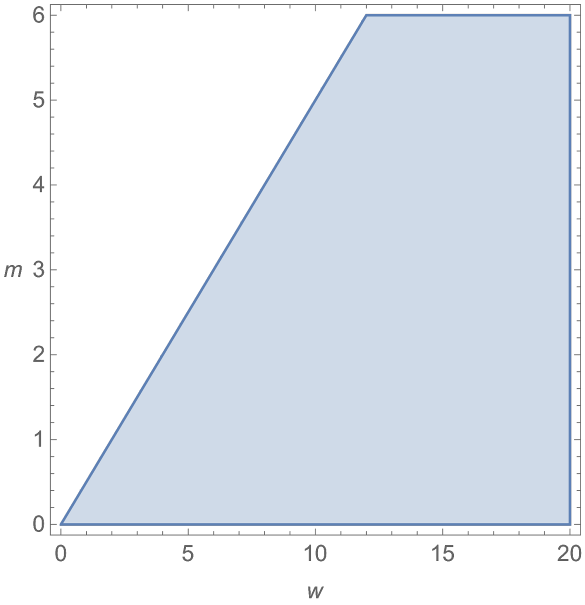
- A machine takes two workers to operate
- Each machine produces 20 chocolates/hour, each worker produces 5 chocolates/hour
- We need to produce at most 100 chocolates/hour
- Each chocolate costs \$10, each worker gets \$40 per hour

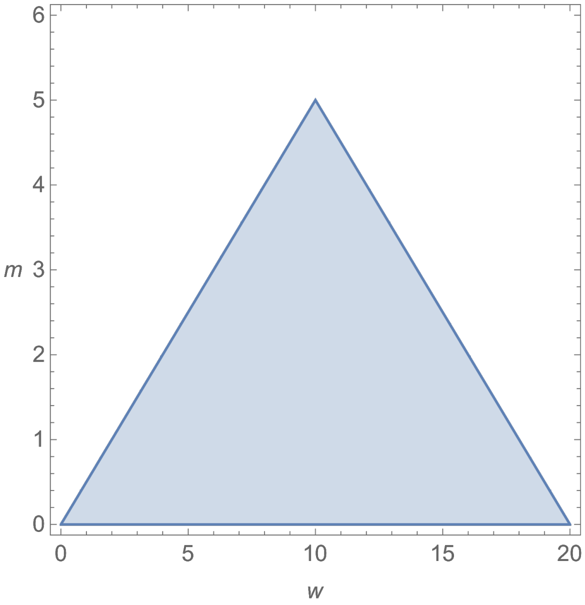
# Workers and Machines



# Two Workers Operate a Machine



# Chocolate Demand



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- Find a linear function $h(a_1, \ldots, a_d)$ s.t.
    - $h(a_1, \ldots, a_d) < 0$ for all spam emails
    - $h(a_1, \ldots, a_d) > 0$ for all ham emails

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- Find real numbers $x_1, \ldots, x_n$ that satisfy linear constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\geq b_2 \\ &\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\geq b_m \end{aligned}$$

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- So that linear objective is maximized

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

# Equivalent Formulations

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- Turn $\leq$ into $\geq$:

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$$Ax \geq b$$

$$\text{maximize } cx = \begin{bmatrix} c_1 & \dots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = c_1x_1 + \dots c_nx_n$$

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- Dantzig's algorithm is "One of top 10 algorithms of the 20th century"

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A linear function takes its maximum and minimum values on vertices

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A linear function takes its maximum and minimum values on vertices

- Start at any vertex
- While there is an adjacent vertex with higher profit
    - Move to that vertex

# Corner Cases

- No solutions

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- Unbounded profit

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