GEMS OF TCS

INTEGER LINEAR PROGRAMMING

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AVOIDING SCURVY

• Orange costs $1, grapefruit costs $1; we have budget of $2/day

• Orange weighs 100gm, grapefruit weighs 200gm, we can carry 300gm

• Orange has 50gm of vitamin C, grapefruit has 75gm of vitamin C, maximize daily vitamin C intake.
AVOIDING SCURVY. PLOT

\[
\max 2x + 3y
\]

\[
x + y \leq 2
\]

\[
x + 2y \leq 3
\]

\[
x \geq 0
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y \geq 0
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AVOIDING SCURVY. PLOT

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max $2x + 3y$

$x + y \leq 2$

$x + 2y \leq 2.5$

$x \geq 0$

$y \geq 0$
AVOIDING SCURVY II

max \( 2x + 3y \)

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Linear programming

**Input:** A set of linear inequalities $Ax \leq b$.

**Output:** Real solution that optimizes the objective function.
**Integer linear programming**

**Input:** A set of linear inequalities $Ax \leq b$.

**Output:** Integer solution that optimizes the objective function.
Example

\[ x_1 \geq 0.5 \]
\[ -x_1 + 8x_2 \geq 0 \]
\[ -x_1 - 8x_2 \geq -8 \]
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Can be solved efficiently (Lecture 10)

ILP

Find an **integer** solution of a system of linear inequalities

No polynomial algorithm known!
LP
Find a real solution of a system of linear inequalities
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Algorithm for ILP

\[ \text{max } 2x + y \]

\[ 4x + y \leq 33 \]
\[ 3x + 4y \leq 29 \]
\[ x \geq 0 \]
\[ y \geq 0 \]
\[ x, y \in \mathbb{Z} \]
\textbf{Algorithm for ILP}

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Linear Programming

(max: 9 variables)

Optimize: Max

Objective Function: 2x + y

Subject to:
4x + y \leq 33,
3x + 4y \leq 29,
x \geq 0,

and:
y \geq 0

More constraints (optional):

More constraints (optional):

(multiple constr. in a box are allowed)

**www.ordsworks.com** *(constraints separator: ",")*

Global maximum:

\[
\max\{2x + y \mid 4x + y \leq 33 \land 3x + 4y \leq 29 \land x \geq 0 \land y \geq 0\} \approx 17.1538 \quad \text{at} \quad (x, y) \approx (7.92308, 1.30769)
\]
Branching on $x$

Original
$\text{OPT} \approx 17.1538$
BRANCHING ON $x$

Original
$\text{OPT} \approx 17.1538$

$x \leq 7$

$x \geq 8$

Prob 1

Prob 2
\[ \text{max } 2x + y \]

\[ 4x + y \leq 33 \]
\[ 3x + 4y \leq 29 \]
\[ x \geq 0 \]
\[ y \geq 0 \]
\[ x, y \in \mathbb{Z} \]
max \(2x + y\)

\(4x + y \leq 33\)
\(3x + 4y \leq 29\)
\(x \geq 0\)
\(y \geq 0\)
\(x, y \in \mathbb{Z}\)
(max: 9 variables)

Optimize: Max

Objective Function: 2x+y

Subject to:
4x+y<=33, 3x+4y<=29,

x>=0,

and:

y>=0,

More constraints (optional):
x<=7

Global maximum:

\[ \text{max}\{2x + y | 4x + y \leq 33 \land 3x + 4y \leq 29 \land x \geq 0 \land y \geq 0 \land x \leq 7\} = 16 \]

at \((x, y) = (7, 2)\)
BRANCHING ON $x$

Original
$\text{OPT} \approx 17.1538$

$\begin{align*}
x \leq 7 \\
x \geq 8
\end{align*}$

Prob 1
$\text{OPT} = 16$

Prob 2
(max: 9 variables)

Optimize: Max

Objective Function: \( 2x + y \)

Subject to: \( 4x + y \leq 33, \quad 3x + 4y \leq 29, \)
\( x \geq 0, \)
\( y \geq 0, \)

and:

More constraints (optional): \( x \geq 8 \)

More constraints (optional):

(multiple constr. in a box are allowed)

**www.oridsworks.com** *(constraints separator: ",")*

Global maximum:

\[
\max \{ 2x + y \mid 4x + y \leq 33 \land 3x + 4y \leq 29 \land x \geq 0 \land y \geq 0 \land x \geq 8 \} = 17
\]

at \((x, y) = (8, 1)\)
Branching on $x$

Original
$\text{OPT} \approx 17.1538$

$x \leq 7$

Prob 1
$\text{OPT} = 16$

$x \geq 8$

Prob 2
$\text{OPT} = 17$
HEURISTIC ALGORITHMS FOR ILP
Applications
APPLICATIONS

• Scheduling

• Planning

• Networks

• ...
Vertex Covers

- A Vertex Cover of a graph $G$ is a set of vertices $C$ such that every edge of $G$ is connected to some vertex in $C$. 
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A **Minimum Vertex Cover** is a vertex cover of the smallest size.
VERTEX COVERS: EXAMPLES
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VERTEX COVER AS ILP

• Introduce binary variable for every vertex: $x_1, \ldots, x_n$:
  • $x_i = 1$ iff $x_i$ belongs to Vertex Cover
**Vertex Cover as ILP**

• Introduce binary variable for every vertex: $x_1, \ldots, x_n$:
  • $x_i = 1$ iff $x_i$ belongs to Vertex Cover
• $\forall i \in \{1, \ldots, n\}$, $0 \leq x_i \leq 1$, $x_i \in \mathbb{Z}$
VERTEX COVER AS ILP

• Introduce binary variable for every vertex: $x_1, \ldots, x_n$:
  - $x_i = 1$ iff $x_i$ belongs to Vertex Cover
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• $\min \sum_i x_i$
VERTEX COVER AS ILP

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  • $x_i = 1$ iff $x_i$ belongs to Vertex Cover
  • $\forall i \in \{1, \ldots, n\}$, $0 \leq x_i \leq 1$, $x_i \in \mathbb{Z}$

• $\min \sum x_i$

• For every edge $(u, v)$ in the graph: $x_u + x_v \geq 1$
```python
import networkx as nx
from mip import *

# Create a graph

g = nx.Graph()
g.add_edges_from([(1, 2), (1, 3), (1, 5), (1, 6), (2, 5), (2, 0),
                   (3, 4), (3, 5), (3, 6), (5, 6), (7, 0)])

m = Model()
n = g.number_of_nodes()
x = [m.add_var(var_type=BINARY) for i in range(n)]

for u, v in g.edges():
    m += x[u] + x[v] >= 1

m.objective = minimize(xsum(x[i] for i in range(n)))
m.optimize()

selected = [i for i in range(n) if x[i].x >= 0.99]
print("selected items: {\{\}}\".format(selected))
```
N Queens

Is it possible to place $n$ queens on an $n \times n$ board such that no two of them attack each other?
N QUEENS AS ILP

• $n^2$ 0/1-variables: for $0 \leq i, j < n$, $x_{ij} = 1$ iff queen is placed into cell $(i, j)$
N QUEENS AS ILP

- \( n^2 \) 0/1-variables: for \( 0 \leq i, j < n \), \( x_{ij} = 1 \) iff queen is placed into cell \((i, j)\)
- For \( 0 \leq i < n \), \( i \)th row contains \( = 1 \) queen:
\[
\sum_{j=1}^{n} x_{ij} = 1.
\]
- For \( 0 \leq i < n \), \( i \)th column contains \( = 1 \) queen:
\[
\sum_{j=1}^{n} x_{ij} = 1.
\]
- Each diagonal contains \( \leq 1 \) queen:
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} \leq \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} \leq 1.
\]
N QUEENS AS ILP

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N Queues as ILP

- $n^2$ 0/1-variables: for $0 \leq i, j < n$, $x_{ij} = 1$ iff queen is placed into cell $(i, j)$
- For $0 \leq i < n$, $i$th row contains $= 1$ queen:
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- For $0 \leq j < n$, $j$th column contains $= 1$ queen:
  \[
  \sum_{j=1}^{n} x_{ij} = 1.
  \]
- Each diagonal contains $\leq 1$ queen:
  \[
  \sum_{i=1}^{n} \sum_{j=1: i-j=k}^{n} x_{ij} \leq 1; \quad \sum_{i=1}^{n} \sum_{j=1: i+j=k}^{n} x_{ij} \leq 1
  \]