GEMS OF TCS

GÖDEL’S INCOMPLETENESS

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March 11, 2021
Godel’s Incompleteness Theorem

1931 Gödel
1936 Turing
(Lecture 13)

\[ x = y \land y = 2 \]

\[ x = 2 \]
Axiomatization of Math

- Find a set of simple and obvious axioms
Axiomatization of Math

- Find a set of simple and obvious axioms

- Any proof could be (in principle) traced back to this set of axioms
Euclid’s Axioms

- For any pair of distinct points, there is exactly one line connecting them
Euclid’s Axioms

• For any pair of distinct points, there is exactly one line connecting them
• Any line segment can be extended to an infinite line
Euclid’s Axioms

- For any pair of distinct points, there is exactly one line connecting them
- Any line segment can be extended to an infinite line
- For any pair of distinct points, there is exactly one circle centered at the first and touching the second
Euclid’s Axioms

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- For any pair of distinct points, there is exactly one circle centered at the first and touching the second
- All right angles are equal to one another
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- [The Parallel Postulate] Given a line $L$ and a point $x$, there is exactly one line parallel to $L$ that passes through $x$
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- For any pair of distinct points, there is exactly one line connecting them.
- Any line segment can be extended to an infinite line.
- For any pair of distinct points, there is exactly one circle centered at the first and touching the second.
- All right angles are equal to one another.
- [The Parallel Postulate] Given a line $L$ and a point $x$, there is exactly one line parallel to $L$ that passes through $x$. 
PEANO ARITHMETIC

• 0 is a natural number
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• \( \forall x, \; x = x \)

• If \( x = y \), then \( y = x \)

• If \( x = y \) and \( y = z \), then \( x = z \)

• ...
PEANO ARITHMETIC

- 0 is a natural number
- ∀x, x = x
- If x = y, then y = x
- If x = y and y = z, then x = z
- ...
- ∀x, y, x = y iff Next(x) = Next(y)
- If x is a natural number, then Next(x) is a natural number
- There is no natural x s.t. Next(x) = 0

Next(x) = x + 1
Peano Arithmetic

- 0 is a natural number
- ∀x, x = x
- If x = y, then y = x
- If x = y and y = z, then x = z
- ...  
- ∀x, y, x = y iff Next(x) = Next(y)
- If x is a natural number, then Next(x) is a natural number
- ...  
- ∀x, y, x + Next(y) = Next(x + y)
PEANO ARITHMETIC

• 0 is a natural number
• $\forall x, x = x$
• If $x = y$, then $y = x$
• If $x = y$ and $y = z$, then $x = z$
• $\ldots$
• $\forall x, y, x = y$ iff $\text{Next}(x) = \text{Next}(y)$
• If $x$ is a natural number, then $\text{Next}(x)$ is a natural number
• $\ldots$
• $\forall x, y, x + \text{Next}(y) = \text{Next}(x + y)$
• $\forall x, y, x \cdot \text{Next}(y) = x \cdot y + x$
Peano Arithmetic

• 0 is a natural number
• ∀x, x = x
• If x = y, then y = x
• If x = y and y = z, then x = z
• ...
• ∀x, y, x = y iff Next(x) = Next(y)
• If x is a natural number, then Next(x) is a natural number
• ...
• ∀x, y, x + Next(y) = Next(x + y)
• ∀x, y, x · Next(y) = x · y + x
• Induction
NAIVE SET THEORY

Cantor, Dedekind, ---

• Set

• Membership in a Set

• Empty Set $\emptyset$

• Equality $A = B$ iff they contain same elements
Russell’s Paradox

$S = \text{set of all sets that don't contain themselves as an element}$

**Case 1:** $S$ contains itself as an element

$\Rightarrow S$ cannot contain $S$ as an element

**Case 2:** $S$ doesn't contain itself as an element

$\Rightarrow S$ must contain $S$ as an element.
\[ T = "S contains itself" \]
\[ \neg T = "S doesn't contain itself" \]

We can prove \( T \) and \( \neg T \).

\[ T \text{ is true } \Rightarrow \neg T \text{ OR } "I'm the Pope" \text{ is true} \]
\[ \neg T \text{ is true} \]
\[ "I'm the Pope" \]
Russell’s Paradox

\[ S = \text{contains all sets that don't contain themselves as an element} \]

The barber is the "one who shaves all those, and those only, who do not shave themselves". The question is, does the barber shave himself?
Bertrand & Whitehead objects can't contain objects of the same type

Took 379 pages to prove $1+1=2$

Proof assistants (Coq, Isabelle, ...)

1. Verify formal proofs
2. Help us to prove theorems
Zermelo & Fraenkel

Set Theory

$\downarrow$

Integers

$\downarrow$

Reals

$\downarrow$

Tuples

$\downarrow$

Graphs

$(x, y) = 2^x \cdot 3^y$
Godel’s Incompleteness Theorem

Any attempt to axiomatize all of mathematics is guaranteed to fail
Halting Problem

- Function HALT is defined as follows.
HALTING PROBLEM

• Function HALT is defined as follows.
  • The first input is algorithm A
Halting Problem

• Function HALT is defined as follows.
  • The first input is algorithm A
  • The second input is string x
HALTING PROBLEM

• Function HALT is defined as follows.
  • The first input is algorithm $A$
  • The second input is string $x$
  • $\text{HALT}(A, x) = 1$ if $A$ halts on input $x$
Halting Problem

- Function HALT is defined as follows.
  - The first input is algorithm $A$
  - The second input is string $x$
  - $\text{HALT}(A, x) = 1$ if $A$ halts on input $x$
  - $\text{HALT}(A, x) = 0$ if $A$ enters infinite loop on input $x$
Halting Problem

- Function HALT is defined as follows.
  - The first input is algorithm A
  - The second input is string $x$
  - $\text{HALT}(A, x) = 1$ if A halts on input $x$
  - $\text{HALT}(A, x) = 0$ if A enters infinite loop on input $x$
- HALT is undecidable (Lecture 13)
Gödel’s Incompleteness Theorem

Any formal system is unsound or incomplete.

\[ \text{sound} \equiv \text{proves only true statements} \]
\[ \text{unsound} \equiv \text{proves some false statements} \]
\[ \text{complete} \equiv \text{for every statement } T, \text{ it proves } T \text{ or } \neg T \]
\[ \text{incomplete} \equiv \text{there is a statement } T \text{ s.t. our system cannot prove } T \text{ nor } \neg T. \]

EQ: Any sound system must be incomplete.

Believe ZFC is sound.
Alg. for Halting Problem \((A,x)\)

Brute force all strings
For each strings

- check \(S\) is a proof (in ZFC) that
  \(A(x)\) halts: Output 1 Exit
- check \(S\) is a proof (in ZFC) that
  \(A(x)\) runs forever: Output 0 Exit

Case I: we output 0 instead of 1
  or output 1 instead of 0
  \(A(x)\) doesn't halt, but we have
  a proof that \(A(x)\) halts
  \(\Rightarrow\) Formal system is unsound

Case II: sometimes (for some \(A\) and \(x\))
  Blue algorithm runs forever
  \(\Rightarrow\) Formal system is incomplete
Two examples of statements that cannot be proved or disproved in ZF (ZFC):

- Axiom of Choice
- Continuum Hypothesis

\( |\mathbb{N}| < ? < |\mathbb{R}| \)

\[ \begin{align*}
\text{Gödel's incompleteness theorem} \\
\text{ZFC} \\
T = "\text{ZFC doesn't prove this statement}" \quad (\text{T})
\end{align*} \]