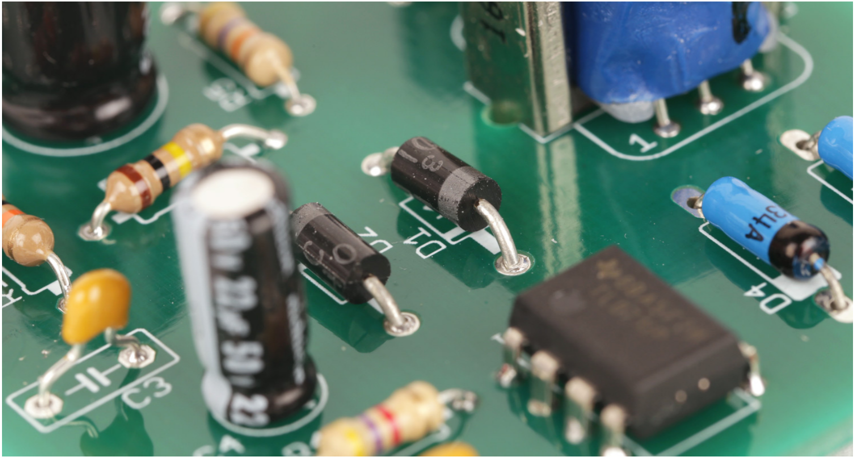
Gems of TCS

Circuit Complexity II

Sasha Golovnev

October 25, 2023


C3
D1
D4

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- If P=NP, then all search problems can be solved in polynomial time.
- If P≠NP, then there exist search problems that cannot be solved in polynomial time.

Boolean Circuits

$$f\colon \{0,1\}^n \to \{0,1\}$$

$$\begin{aligned}
g_1 &= \neg x_1 \\
g_2 &= x_2 \wedge x_3 \\
g_3 &= g_1 \vee g_2 \\
g_4 &= g_2 \vee 1 \\
g_5 &= g_3 \wedge g_4
\end{aligned}$$

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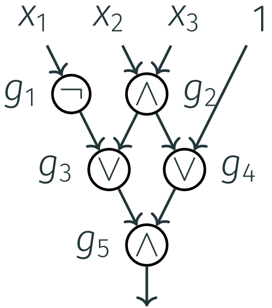

Exponential Bounds

Lower Bound [Sha1949]

Almost all functions of n variables have circuit size

$$\geq 2^n/n$$

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Upper Bound [Lup1958]

Any function can be computed by a circuit of size

$$\leq 2^n/n$$

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for some $h\colon \{0,1\}^{\log n} \to \{0,1\}$, where
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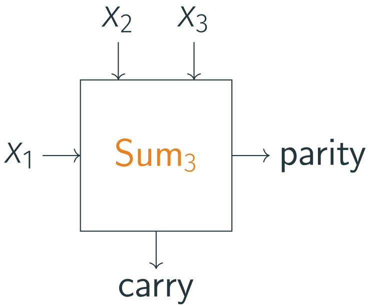

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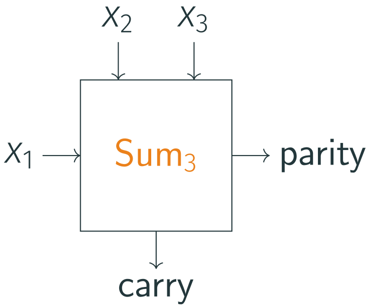

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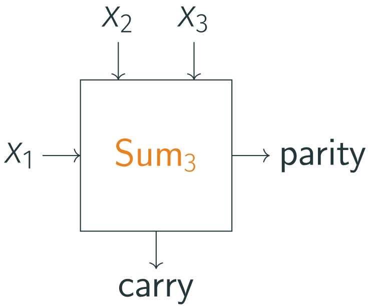

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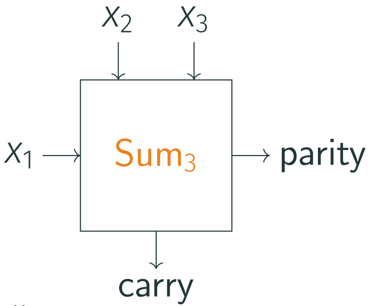

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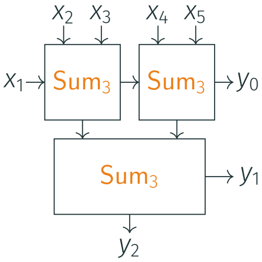

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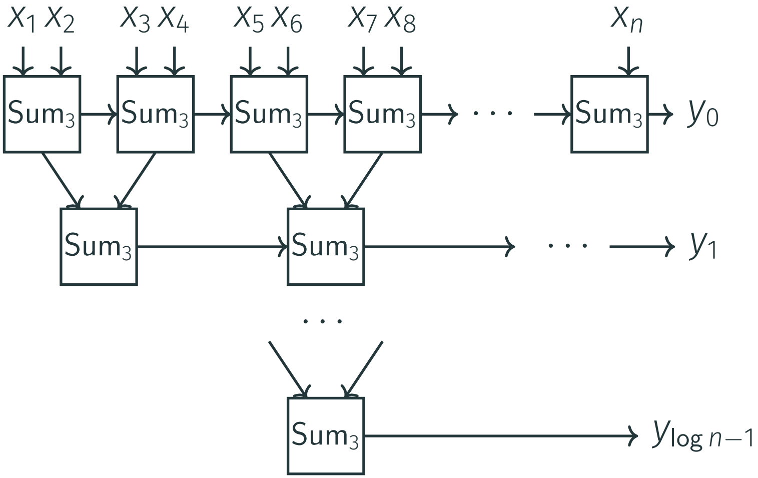

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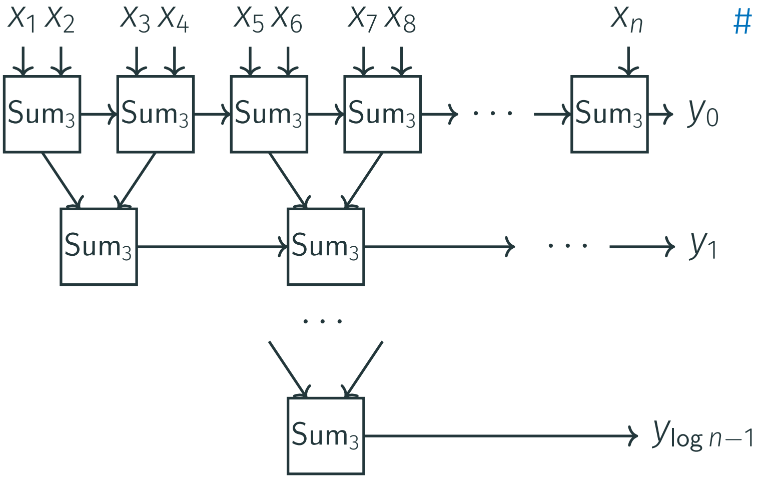

blocks

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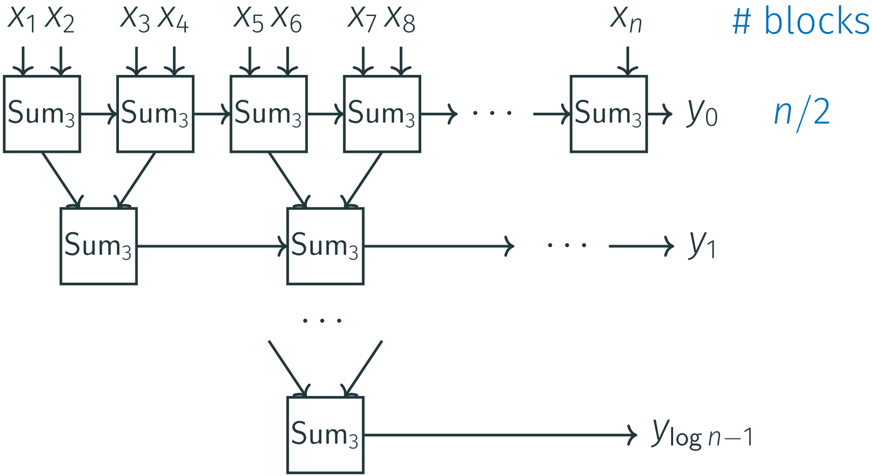

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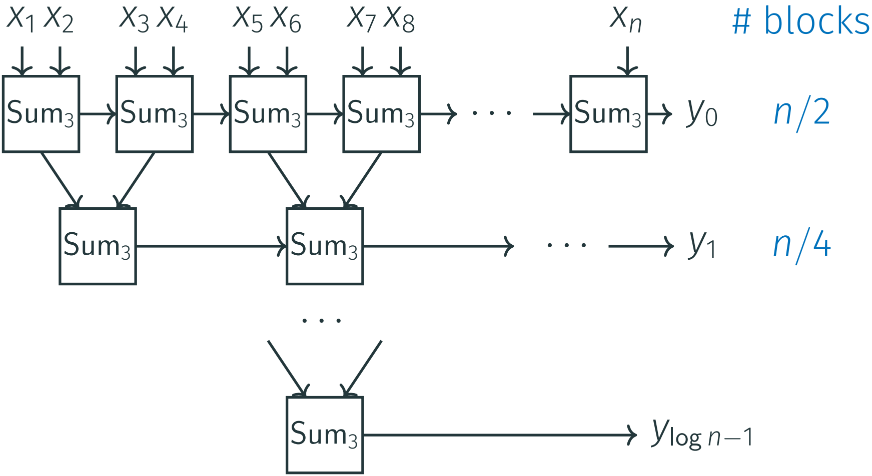

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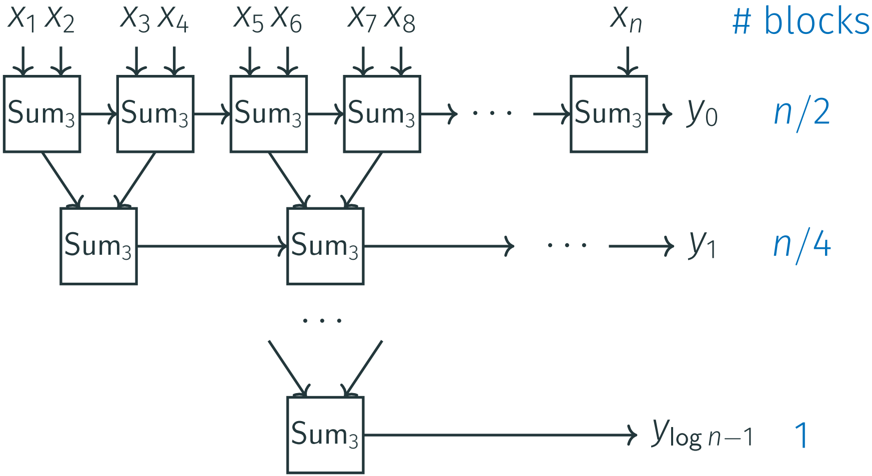

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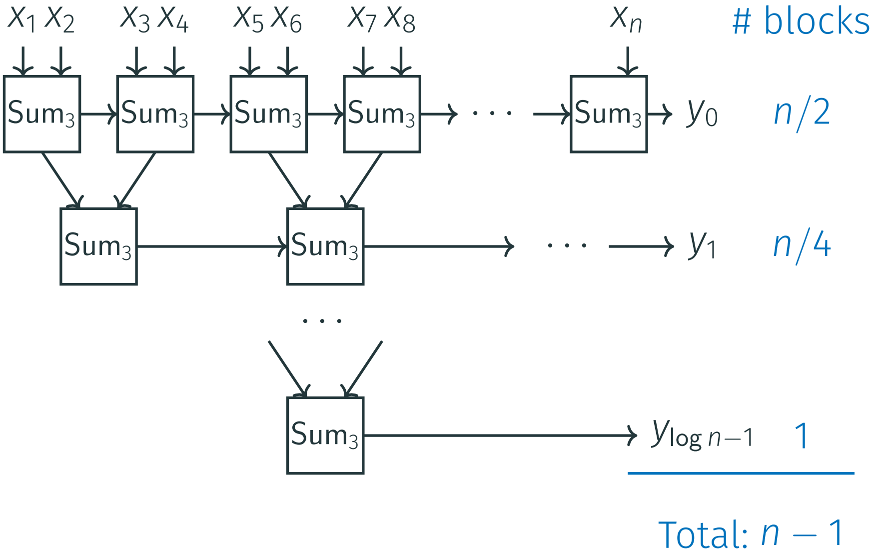

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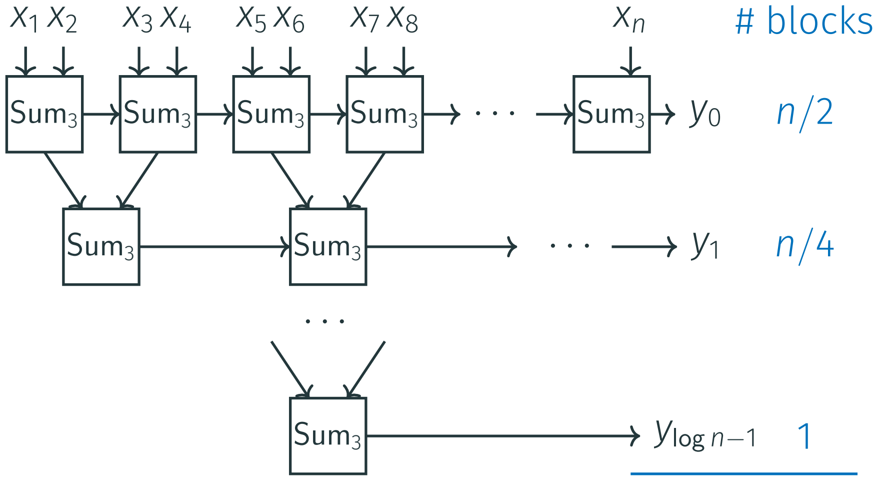

$\mathrm{Size}(\mathrm{Sum}_n) < n \cdot \mathrm{Size}(\mathrm{Sum}_3) = O(n)$

Total: $n - 1$

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$$\mathrm{Size}(h) \leq 10 \cdot 2^{\log n} = O(n)$$

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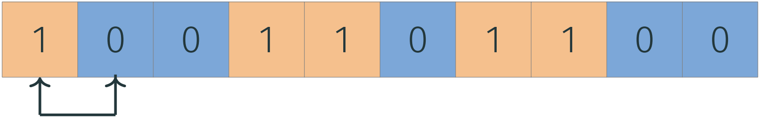

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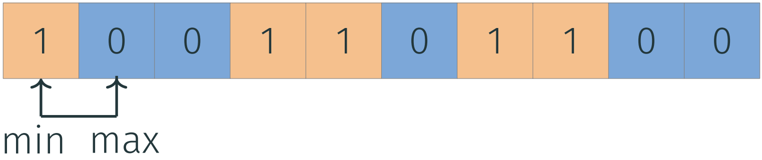

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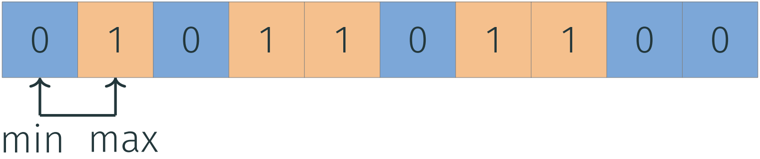

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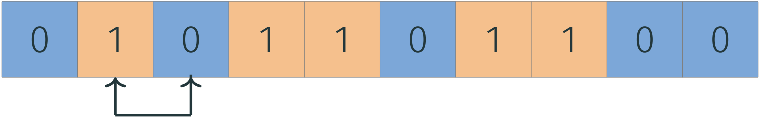

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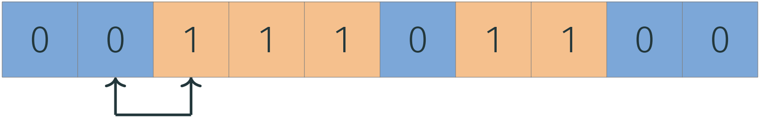

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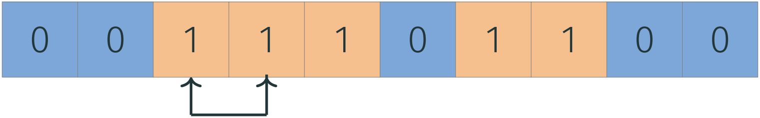

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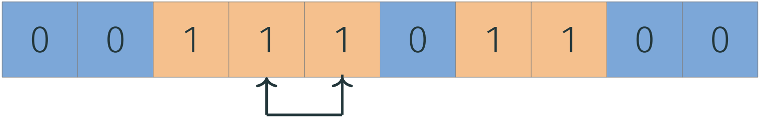

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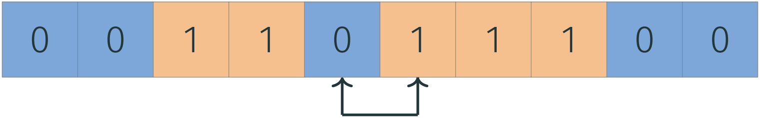

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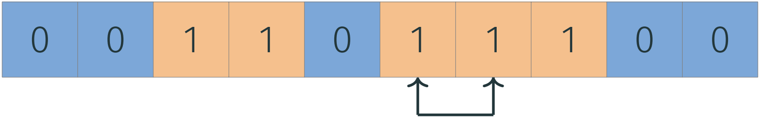

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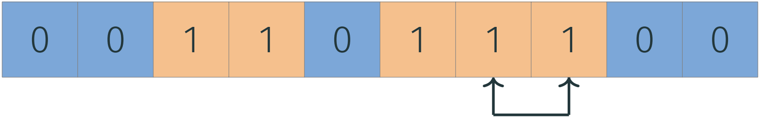

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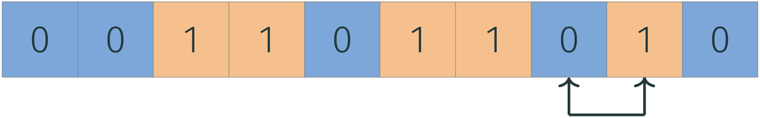

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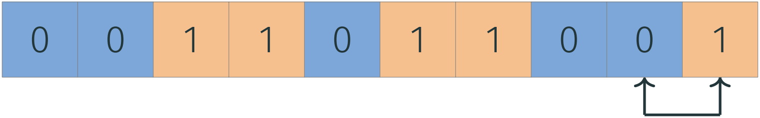

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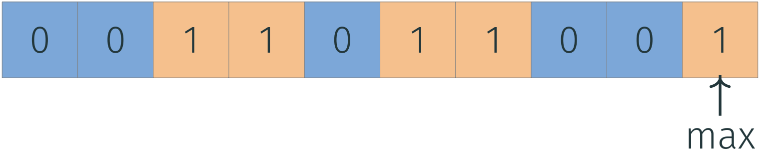

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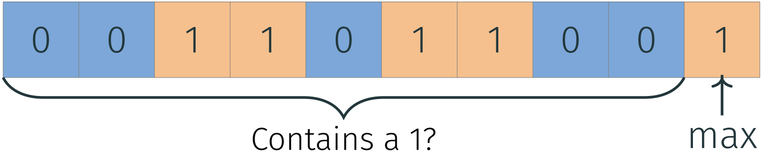
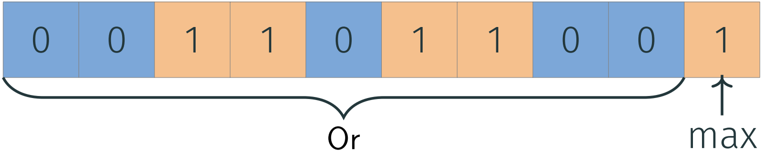
- Two rounds of “Bubble Sort”



Complexity of Threshold

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

- $k = 1$, $\text{Th}_1 = \text{Or}$, $\text{Size}(\text{Th}_1) = n - 1$
- $k = 2$, $\text{Size}(\text{Th}_2) = O(n)$
- $\text{Size}(\text{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



Th_2. Upper Bound

$$\boxed{\begin{array}{ccc} X_1 & \cdots & X_{\sqrt{n}} \\ \vdots & & \vdots \\ X_{n-\sqrt{n}+1} & & X_n \end{array}}$$

Th_2. Upper Bound

$$\boxed{\begin{array}{ccc} x_1 & \cdots & x_{\sqrt{n}} \\ \vdots & & \vdots \\ x_{n-\sqrt{n}+1} & & x_n \end{array}}$$

$\mathrm{Th}_2(x_1, \ldots, x_n) = 1$ iff

Th_2. Upper Bound

$$\begin{array}{|ccc|}\hline x_1 & \cdots & x_{\sqrt{n}} \\ \vdots & & \vdots \\ x_{n-\sqrt{n}+1} & & x_n \\ \hline \end{array}$$

$\mathrm{Th}_2(x_1, \ldots, x_n) = 1$ iff

there are two cols with 1s

OR

there are two rows with 1s

Th_2. Upper Bound

$$
\begin{array}{r|ccc|}
\hline
y_1 = \mathrm{Or} & x_1 & \cdots & x_{\sqrt{n}} \\
y_2 = \mathrm{Or} & & & \\
\vdots & \vdots & & \vdots \\
y_{\sqrt{n}} = \mathrm{Or} & x_{n-\sqrt{n}+1} & & x_n \\
\hline
\end{array}
$$

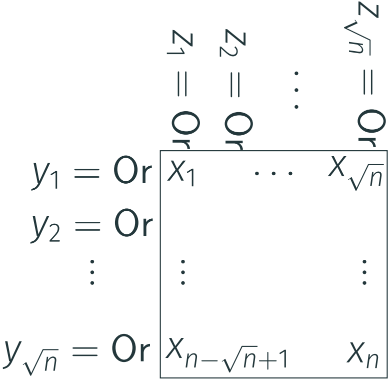
$\mathrm{Th}_2(x_1, \ldots, x_n) = 1$ iff

there are two cols with 1s

OR

there are two rows with 1s

Th_2. Upper Bound



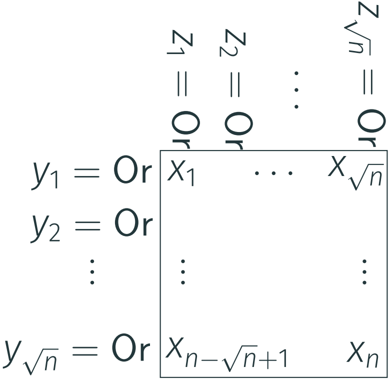
$\mathrm{Th}_2(x_1, \ldots, x_n) = 1$ iff

there are two cols with 1s

OR

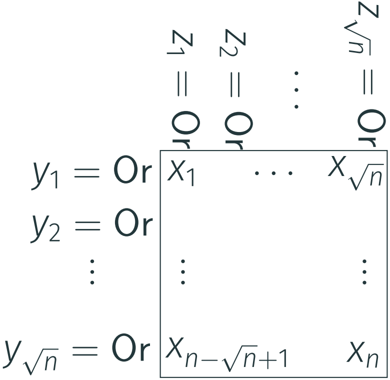
there are two rows with 1s

Th_2. Upper Bound



$$\mathrm{Th}_2(x_1, \ldots, x_n) = \mathrm{Th}_2(y_1, \ldots, y_{\sqrt{n}}) \ \mathrm{Or} \ \mathrm{Th}_2(z_1, \ldots, z_{\sqrt{n}})$$

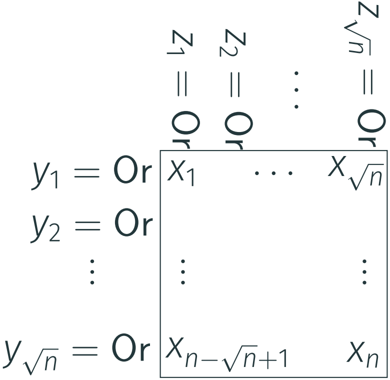
Th_2. Upper Bound



$$\mathrm{Th}_2(x_1, \ldots, x_n) = \mathrm{Th}_2(y_1, \ldots, y_{\sqrt{n}}) \;\; \mathrm{Or} \;\; \mathrm{Th}_2(z_1, \ldots, z_{\sqrt{n}})$$

$$\mathrm{Size}(\mathrm{Th}_2(n)) \leq 2n + 2\,\mathrm{Size}(\mathrm{Th}_2(\sqrt{n}))$$

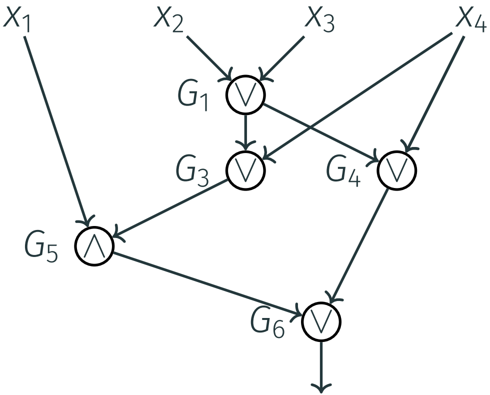
Th_2. Upper Bound



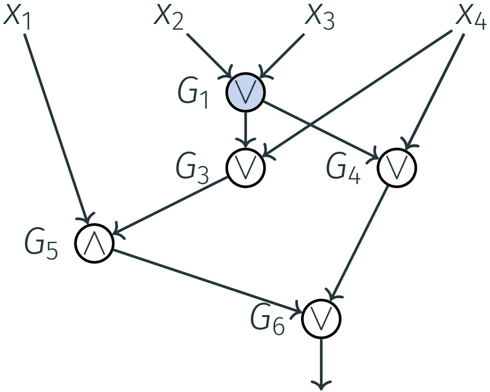
$$\mathrm{Th}_2(x_1, \ldots, x_n) = \mathrm{Th}_2(y_1, \ldots, y_{\sqrt{n}}) \;\; \mathrm{Or} \;\; \mathrm{Th}_2(z_1, \ldots, z_{\sqrt{n}})$$

$$\mathrm{Size}(\mathrm{Th}_2(n)) \leq 2n + 2\,\mathrm{Size}(\mathrm{Th}_2(\sqrt{n})) \leq 2n + O(\sqrt{n})$$

Th_2. Lower Bound

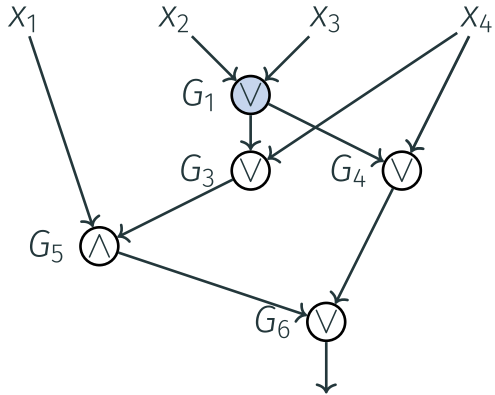


Th_2. Lower Bound



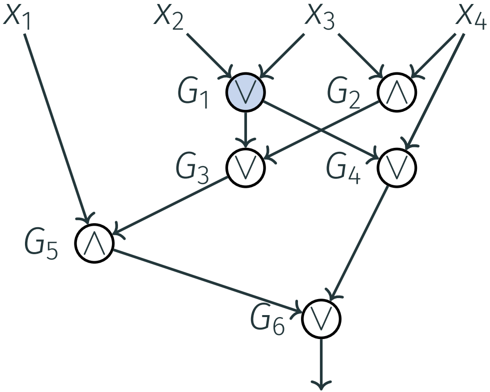
Th_2. Lower Bound

Case I:



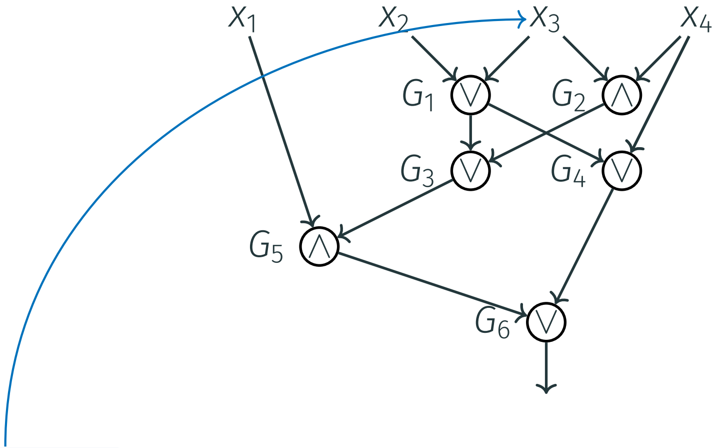
Th_2. Lower Bound

Case II:



Th_2. Lower Bound

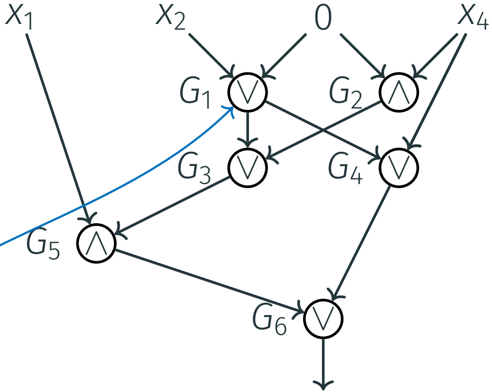
Case II:



assign $x_3 = 0$

Th_2. Lower Bound

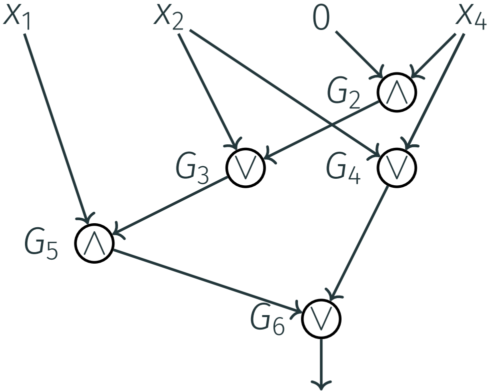
Case II:



G_1 now computes x_2

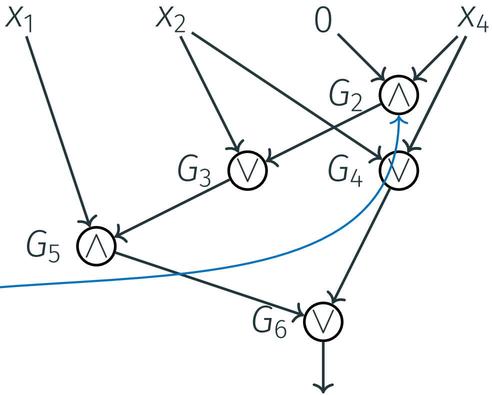
Th_2. Lower Bound

Case II:



Th_2. Lower Bound

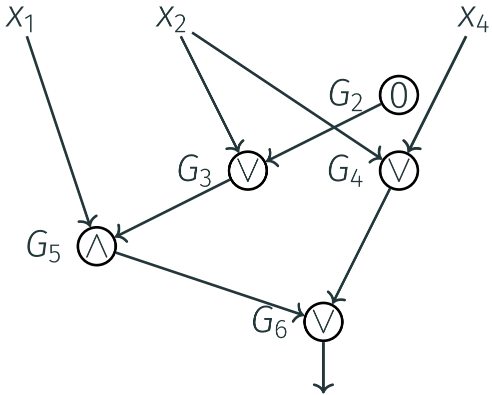
Case II:



$G_2 = 0$

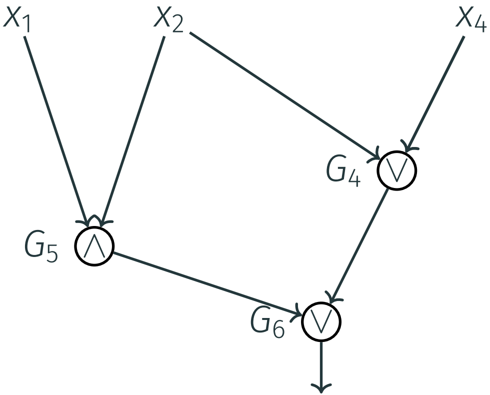
Th$_2$. Lower Bound

Case II:

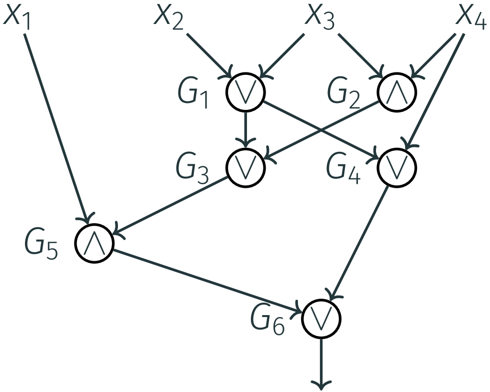


Th_2. Lower Bound

Case II:

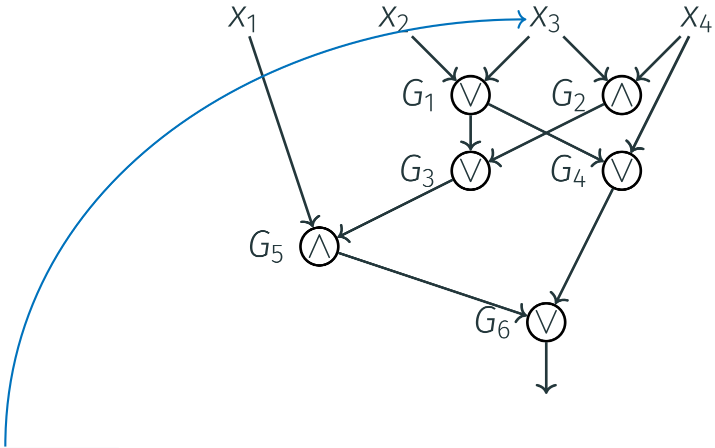


Th_2. Lower Bound



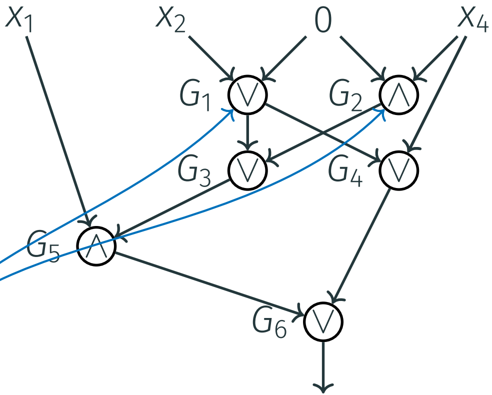
we start with circuit for $\mathbf{Th}_2^n$

Th$_2$. Lower Bound



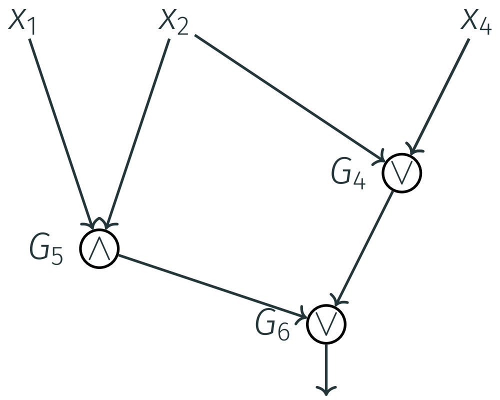
assign $x_3 = 0$

Th$_2$. Lower Bound



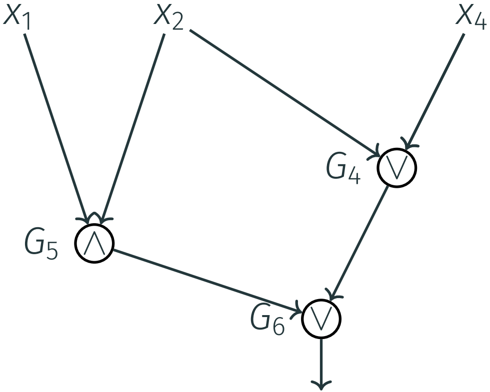
eliminate at least 2 gates

Th_2. Lower Bound



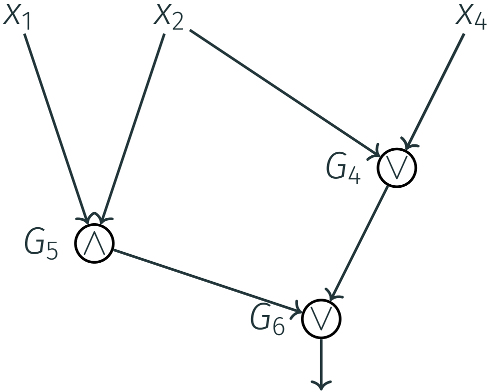
get a circuit for $\mathbf{Th}_2^{n-1}$

Th_2. Lower Bound



$\text{Size}(\text{Th}_2^n) \geq 2 + \text{Size}(\text{Th}_2^{n-1})$

Th_2. Lower Bound



$$\mathrm{Size}(\mathrm{Th}_2^n) \geq 2 + \mathrm{Size}(\mathrm{Th}_2^{n-1}) \geq 2n - O(1)$$