PREVIOUSLY...

- Exact Algorithms
- Randomized Algorithms
- Approximate Algorithms

Today: More examples
PREVIOUSLY...

• Exact Algorithms
• Randomized Algorithms
• Approximate Algorithms

• Today: More examples
Map Coloring
SOUTH AMERICA
THE LAND OF OZ
Theorem [Appel, Haken, 1976]

Every map can be colored with 4 colors.
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- Proved using a computer.

Theorem [Weak Version]
Every map can be colored with 6 colors.
## Four Color Theorem

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### Four Color Theorem

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SIX COLOR THEOREM

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- Induction on the number of countries $n$.  
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- Induction assumption. All maps with $k$ countries can be colored with 6 colors.
Theorem [Weak Version]

Every map can be colored with 6 colors.

- **Induction** on the number of countries $n$.
- **Base case.** $n \leq 6$: can color with 6 colors.
- **Induction assumption.** All maps with $k$ countries can be colored with 6 colors.
- **Induction step.** We’ll show that any map with $k + 1$ countries can be colored with 6 colors.
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SIX COLOR THEOREM. PROOF

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Graph Coloring
Graph Coloring

- A graph coloring is a coloring of the graph vertices such that no pair of adjacent vertices share the same color.
Graph Coloring

- A graph coloring is a coloring of the graph vertices s.t. no pair of adjacent vertices share the same color.

- The chromatic number $\chi(G)$ of a graph $G$ is the smallest number of colors needed to color the graph.
CHROMATIC NUMBER

Chromatic number is 3
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The chromatic number of $K_n$ is $n$. 
For $n > 1$, the chromatic number of $P_n$ is 2.
For even $n$, the chromatic number of $C_n$ is 2.
For odd $n > 2$, the chromatic number of $C_n$ is 3.
The chromatic number of a bipartite graph (with at least 1 edge) is 2.
Applications
EXAM SCHEDULE

• Each student takes an exam in each of her courses
• All students in one course take the exam together
• One student cannot take two exams per day
• What is the minimum number of days needed for the exams?
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Different stations are allowed to use the same frequency if they are far apart. What is an optimal assignment of frequencies to stations?
Bandwidth allocation

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Other Applications

- Scheduling Problems
- Register Allocation
- Sudoku puzzles
- Taxis scheduling
- ...
Exact Algorithm for Coloring
Dynamic Programming

• Given graph $G$ on $n$ vertices, find $\chi(G)$—minimum number of colors in a valid coloring of $G$. 

Dynamic programming is one of the most powerful algorithmic techniques. ⋆

Rough idea: express a solution for a problem through solutions for smaller subproblems.
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Subproblems

• For a subset of vertices $S \subseteq \{1, \ldots, n\}$ compute $\chi(S)$—the minimum number of colors needed to color vertices $S$
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Consider $S$. For any subset $U \subseteq S$, if there are no edges between vertices from $U$, we can color them all in one color, and use $\chi(S \setminus U)$ to color the rest.

$\chi(S) = \min_U \{1 + \chi(S \setminus U)\}$
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$$\chi(S) = \min_{U \text{ without edges}} 1 + \chi(S \setminus U)$$
Order of Subproblems

- Need to process all subsets $S \subseteq \{1, \ldots, n\}$ in order that guarantees that when computing the value of $\chi(S)$, the values of $\chi(S \setminus U)$ have already been computed.
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• For example, we can process subsets in order of increasing size
\[ \chi(\emptyset) = 0 \]
$\chi(\emptyset) = 0$

for $s$ from 1 to $n$:  
  
  for all $S \subseteq \{1, \ldots, n\}$ of size $s$:  

Algorithm

\( \chi(\emptyset) = 0 \)

for s from 1 to n:

for all \( S \subseteq \{1, \ldots, n\} \) of size s:

for all \( U \subseteq S, \ U \) without edges

\( \chi(S) \leftarrow \min\{\chi(S), \chi(S \setminus U) + 1\} \)
**Algorithm**

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return \( \chi(\{1, \ldots, n\}) \)
Running Time

\[ \chi(\emptyset) = 0 \]

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Randomized Algorithm for 3-Coloring
RANDOMIZED ALGORITHM

• Given a 3-colorable graph, find a 3-coloring
RANDOMIZED ALGORITHM

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• This problem is NP-hard, we’ll give an exponential-time algorithm
RANDOMIZED ALGORITHM

• Forbid one random color at each vertex
RANDOMIZED ALGORITHM

- Forbid one random color at each vertex
- Solve 2-SAT in polynomial time
- Repeat the algorithm \((\frac{3}{2})^n\) times
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Approximate Algorithm for 3-Coloring
Approximate Coloring

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- Given a 3-colorable graph, finding a 3-coloring is NP-hard.

- Given a 3-colorable graph, finding an $n$-coloring is trivial.
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• Given a 3-colorable graph, finding a 3-coloring is $\text{NP}$-hard

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• We’ll see how to find an $O(\sqrt{n})$-coloring in polynomial time
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APPROXIMATE ALGORITHM

While there is vertex \( v \in G \) of degree \( \geq \sqrt{n} \):

Color the neighbors of \( v \) in 2 new colors, remove them from the graph.

All remaining vertices have degree \( < \sqrt{n} \).

Color the rest of the graph using \( \sqrt{n} \) new colors.
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All remaining vertices have degree $< \sqrt{n}$. Color the rest of the graph using $\sqrt{n}$ new colors