Gems of TCS

Graph Coloring Algorithms

Sasha Golovnev

September 18, 2023

Previously...

- Exact Algorithms
- Randomized Algorithms
- Approximate Algorithms

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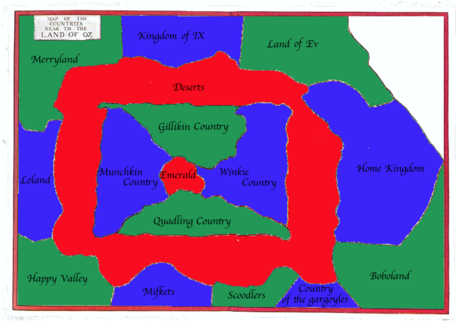
- Randomized Algorithms
- Approximate Algorithms
- Today: More examples

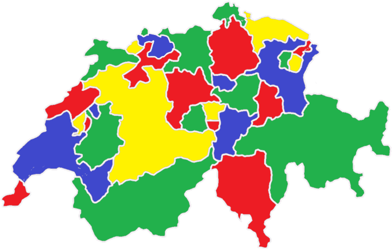
Map Coloring

South America

The Land of Oz



Swiss Cantons



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- Induction on the number of countries n.
- Base case. $n \leq 6$: can color with 6 colors.
- Induction assumption. All maps with k countries can be colored with 6 colors.
- Induction step. We'll show that any map with $k + 1$ countries can be colored with 6 colors.

Six Color Theorem. Proof

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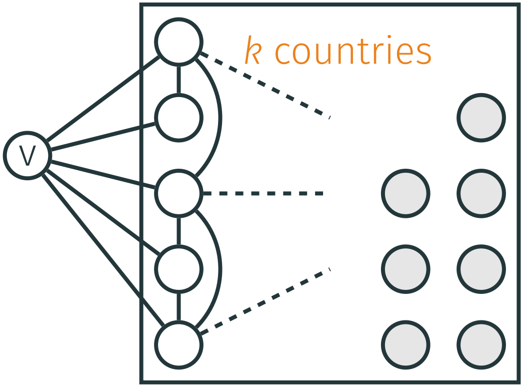

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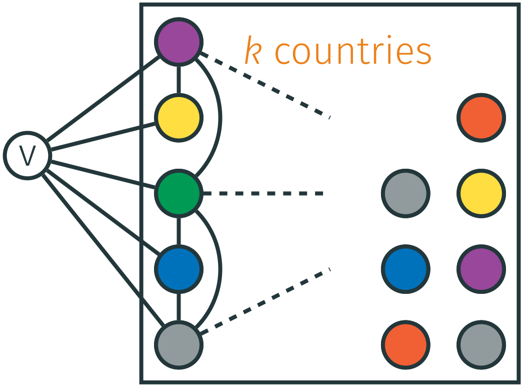

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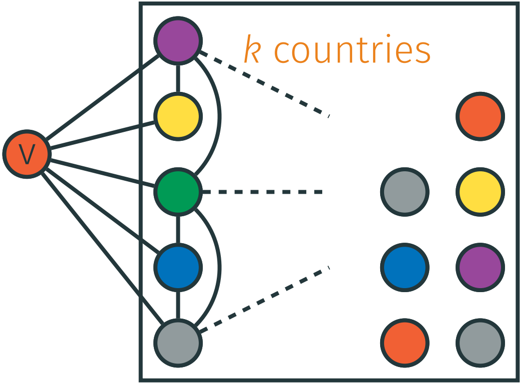

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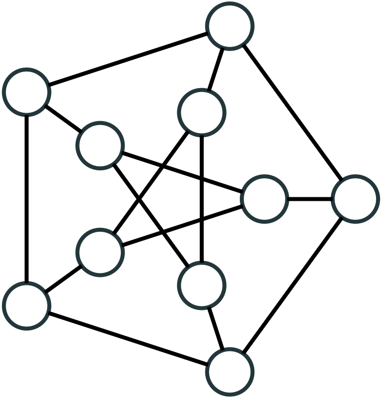
- The chromatic number $\chi(G)$ of a graph G is the smallest number of colors needed to color the graph.

Chromatic Number



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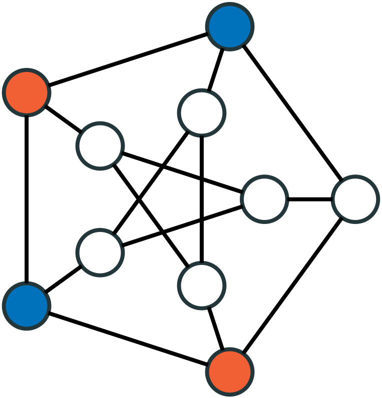

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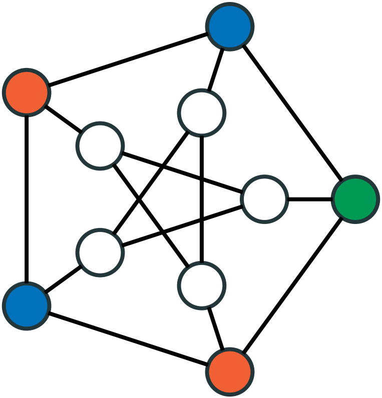

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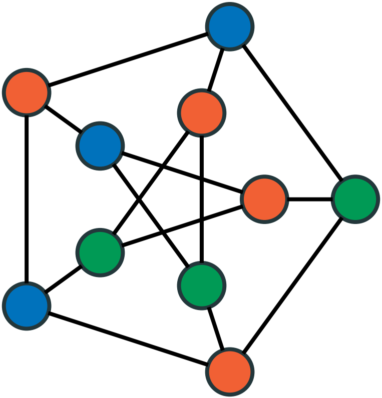
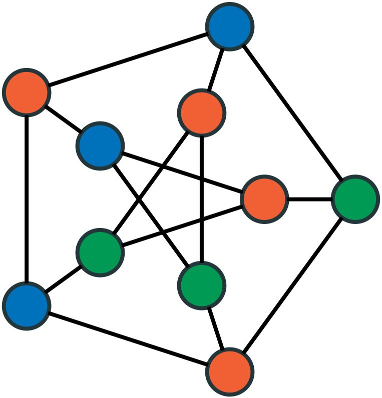


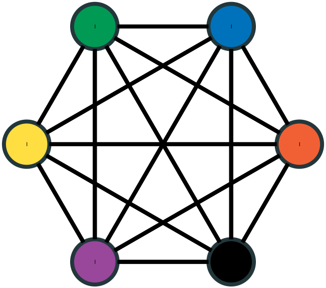
Chromatic Number

Chromatic number is 3



Complete Graphs

The chromatic number of K_n is n.

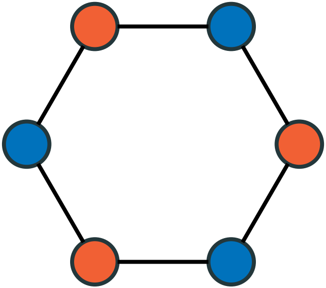


Path Graphs

For $n > 1$, the chromatic number of P_n is 2.

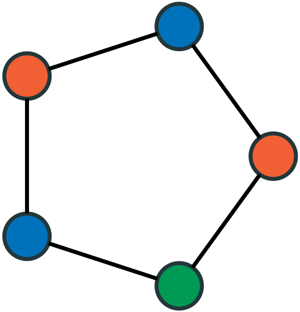
Cycle Graphs

For even n, the chromatic number of C_n is 2.



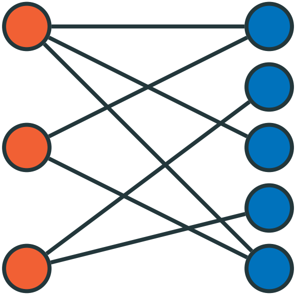
Cycle Graphs

For odd $n > 2$, the chromatic number of C_n is 3.



Bipartite Graphs

The chromatic number of a bipartite graph (with at least 1 edge) is 2.



Applications

Exam schedule

- Each student takes an exam in each of her courses
- All students in one course take the exam together
- One student cannot take two exams per day
- What is the minimum number of days needed for the exams?

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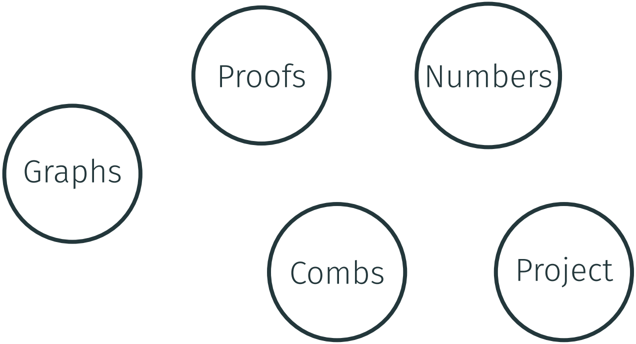

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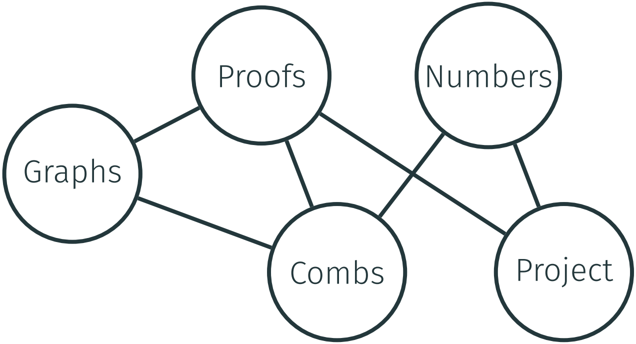

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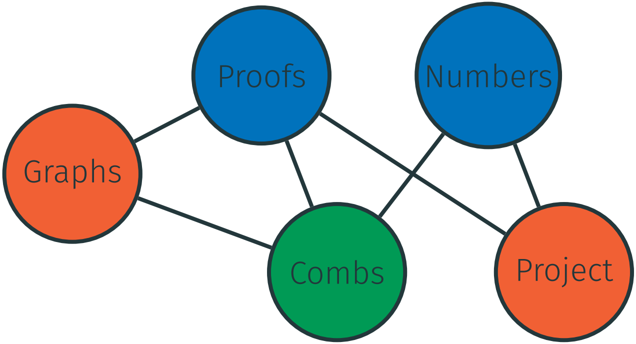
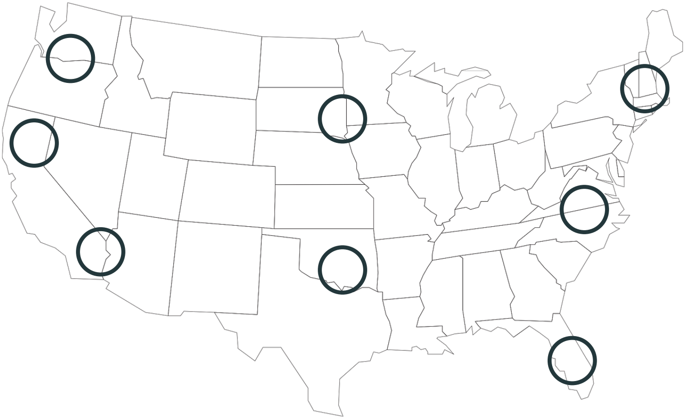


Bandwidth allocation

Different stations are allowed to use the same frequency if they are far apart. What is an optimal assignment of frequencies to stations?



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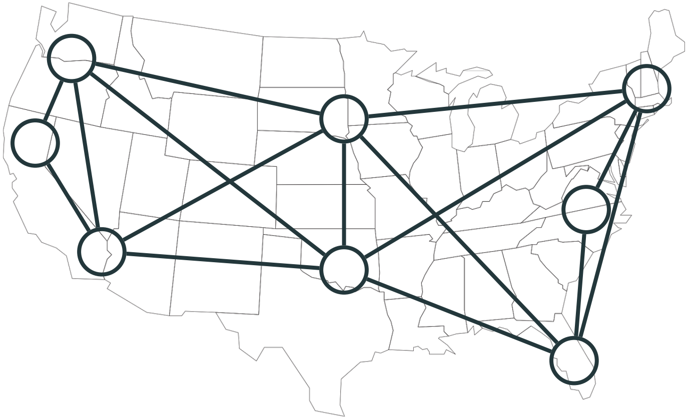

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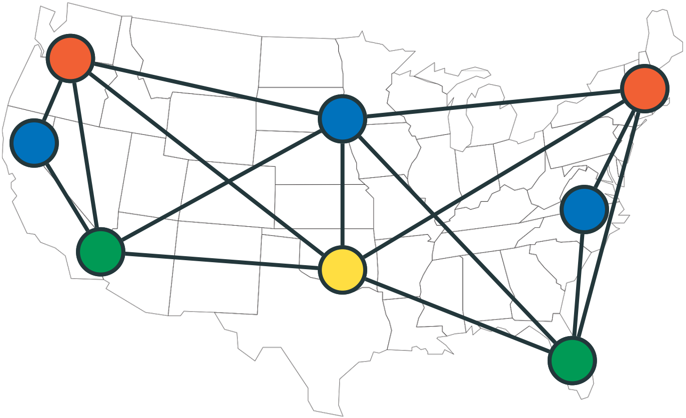

Other Applications

- Scheduling Problems
- Register Allocation
- Sudoku puzzles
- Taxis scheduling
- ...

Exact Algorithm for Coloring

Dynamic Programming

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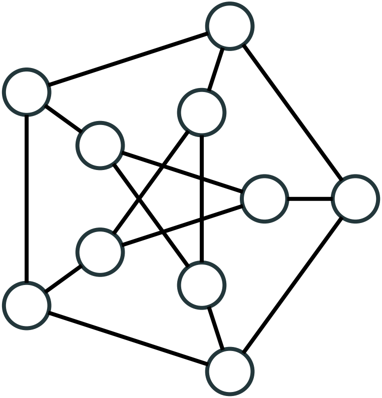
- Dynamic programming is one of the most powerful algorithmic techniques
- Rough idea: express a solution for a problem through solutions for smaller subproblems

Subproblems

- For a subset of vertices $S \subseteq \{1, \ldots, n\}$ compute $\chi(S)$—the minimum number of colors needed to color vertices S

Subproblems



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$$\chi(S) = \min_{U \text{ without edges}} 1 + \chi(S \setminus U)$$

Order of Subproblems

- Need to process all subsets $S \subseteq \{1, \ldots, n\}$ in order that guarantees that when computing the value of $\chi(S)$, the values of $\chi(S \setminus U)$ have already been computed

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- For example, we can process subsets in order of increasing size

Algorithm

$\chi(\emptyset) = 0$

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Running Time

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Randomized Algorithm for 3-Coloring

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- This problem is NP-hard, we'll give an exponential-time algorithm

Randomized Algorithm

- Forbid one random color at each vertex

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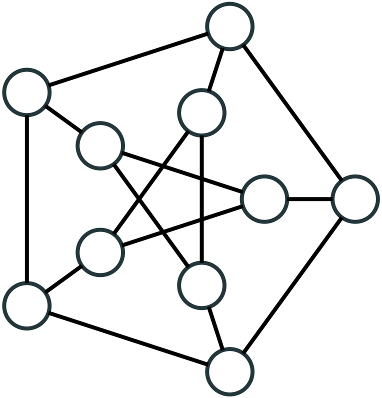

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- Repeat the algorithm $(3/2)^n$ times

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- We'll see how to find an $O(\sqrt{n})$-coloring in polynomial time

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Greedy Coloring

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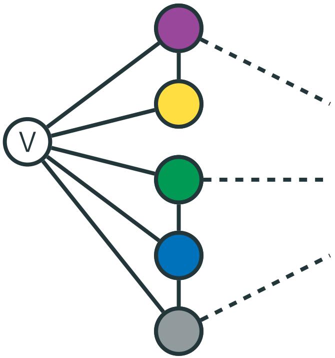

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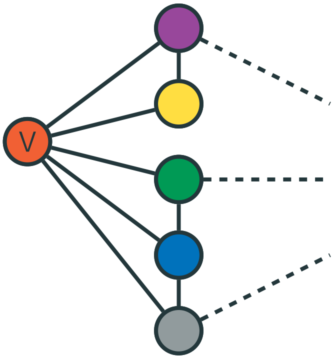

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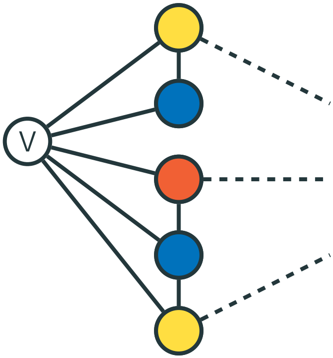

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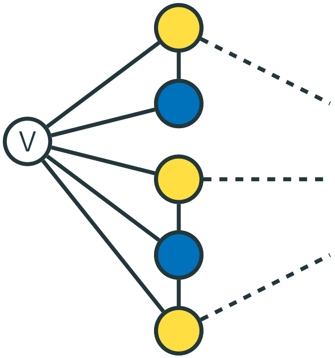

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All remaining vertices have degree $< \sqrt{n}$. Color the rest of the graph using $\sqrt{n}$ new colors

Analysis