Gems of TCS

Heuristic Algorithms

Sasha Golovnev

Semptermber 20, 2023

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Traveling Salesman

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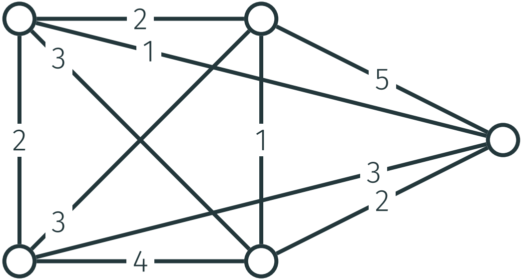
Given a complete weighted graph, find a cycle (or a path) of minimum total weight (length) visiting each node exactly once



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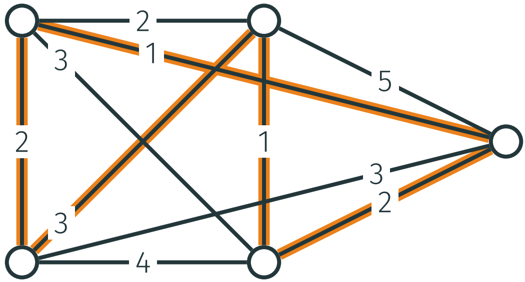

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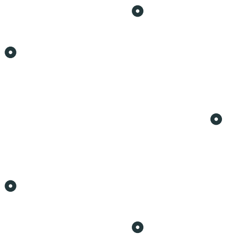

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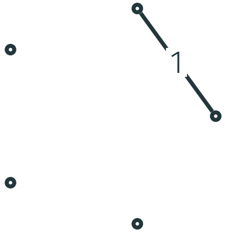

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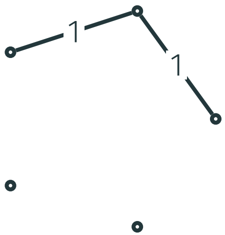

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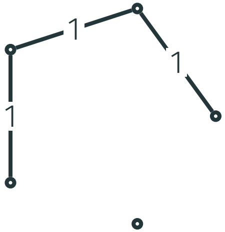

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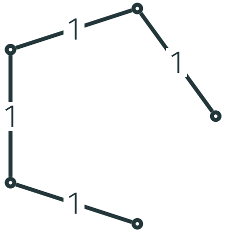

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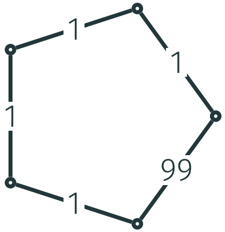
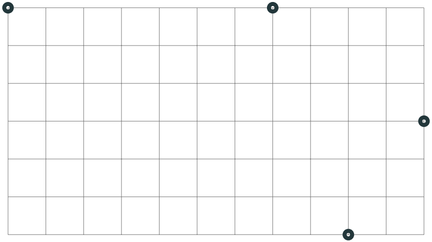


Suboptimal Solution for Euclidean TSP



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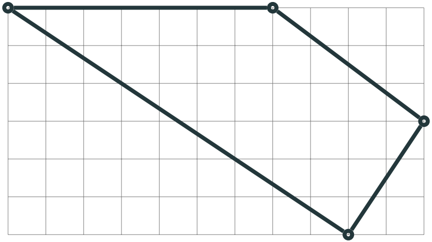

OPT ≈ 26.42

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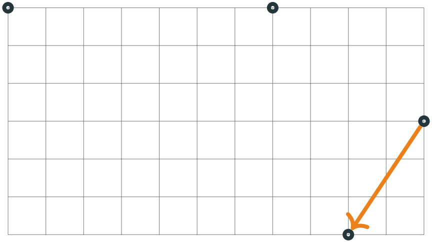

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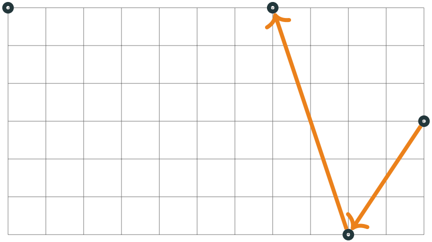

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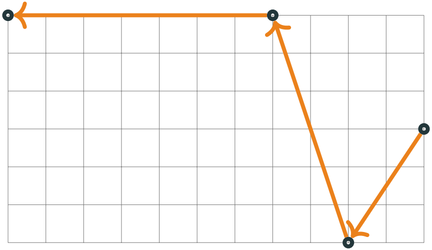

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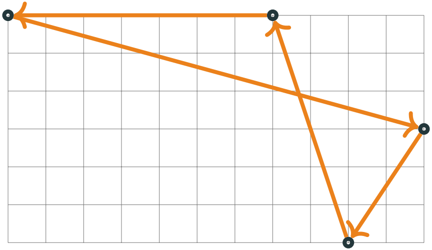

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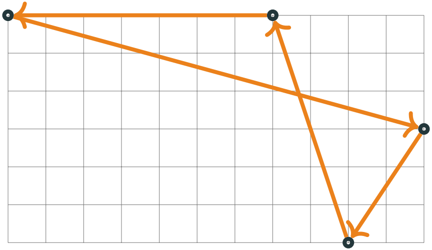

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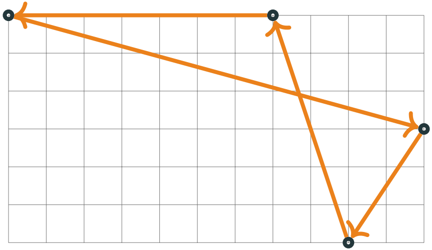

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For Euclidean instances, the resulting cycle is $O(\log n)$-approximate

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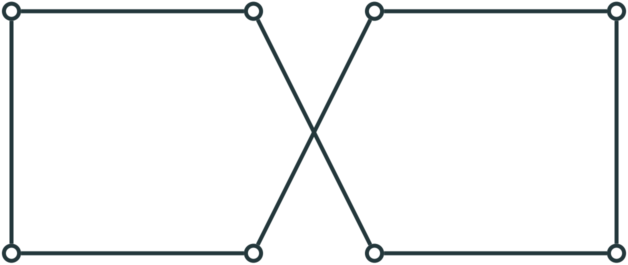
Example

Changing two edges in a suboptimal solution:



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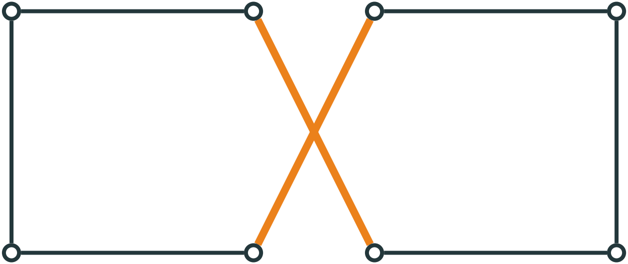

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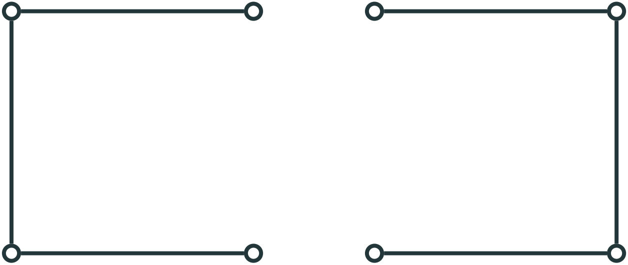

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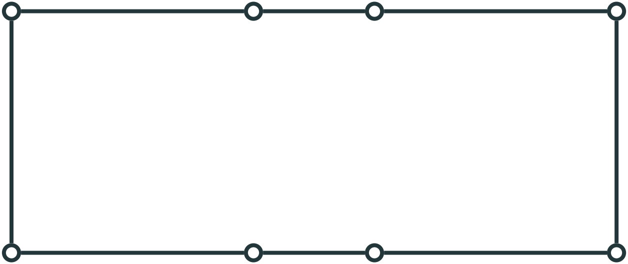

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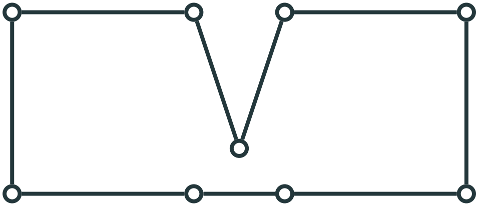

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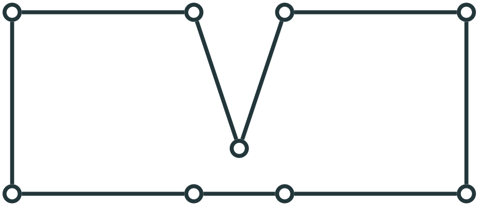

Need to allow changing three edges to improve this solution

Local Search

Local Search with parameter d:

- $s \leftarrow$ some initial solution
- while it is possible to change d edges in s to get a better cycle s':
  - $s \leftarrow s'$
- return s

Properties

- Computes a local optimum instead of a global optimum

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- The larger d, the better the resulting solution and the higher is the running time

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- But works well in practice

Satisfiability

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$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3) \wedge (x_2 \vee \neg x_3)$$

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Backtracking

- Construct a solution piece by piece

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- Backtrack if the current partial solution cannot be extended to a valid solution

Example

$$\boxed{(x_1 \vee x_2 \vee x_3 \vee x_4)(\neg x_1)(x_1 \vee x_2 \vee \neg x_3)(x_1 \vee \neg x_2)(x_2 \vee \neg x_4)}$$

Example

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$$\Big\downarrow{\scriptstyle x_1 = 0}$$

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$$\boxed{(\,)}$$

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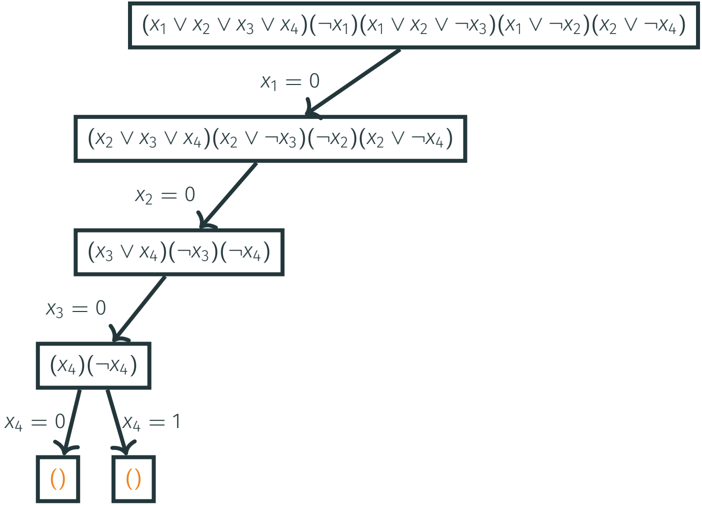
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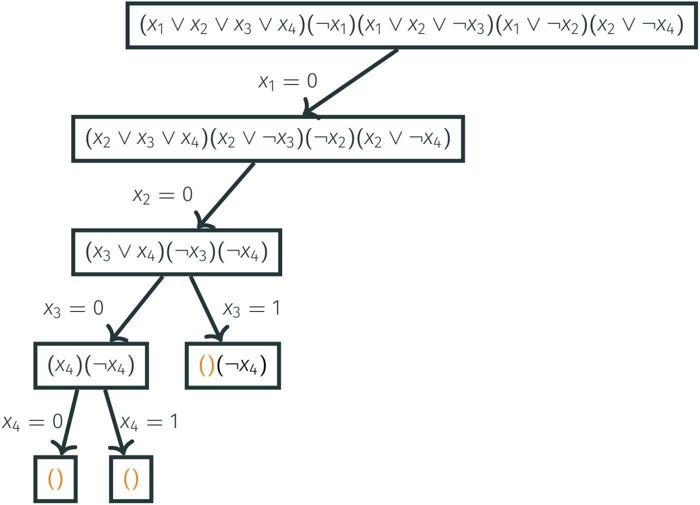
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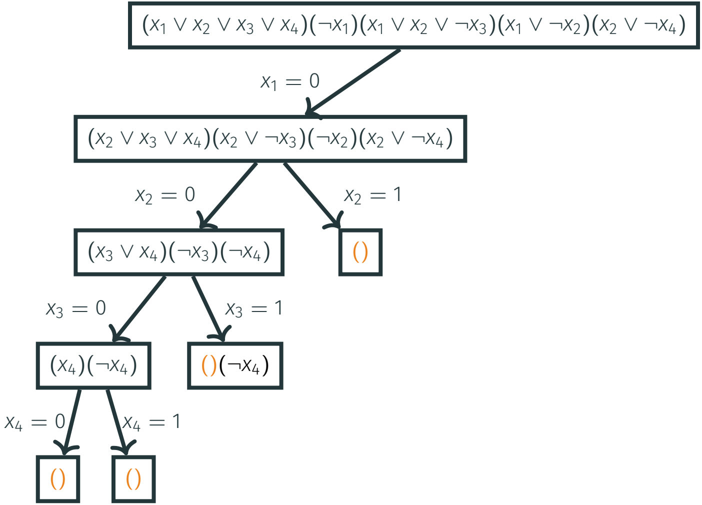

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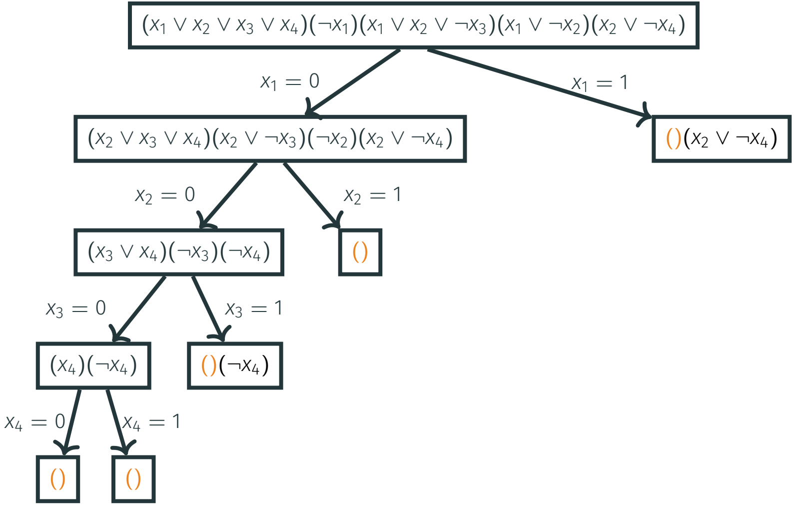
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- SAT-solvers use tricky heuristics to choose a variable to branch on, simplify a formula before branching, and use efficient data structures
- Another commonly used technique is local search

Applications

THE ART OF COMPUTER PROGRAMMING

VOLUME 4 PRE-FASCICLE 6A

A DRAFT OF SECTION 7.2.2.2: SATISFIABILITY

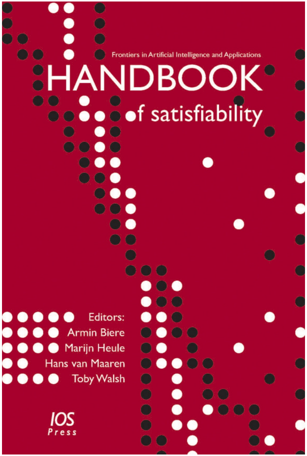
DONALD E. KNUTH *Stanford University*

The Art of Computer Programming

Wow! — Section 7.2.2.2 has turned out to be the longest section, by far, in The Art of Computer Programming. The SAT problem is evidently a "killer app," because it is key to the solution of so many problems. Consequently I can only hope that my lengthy treatment does not also kill off my faithful readers!

Donald Knuth

SAT Handbook



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  `http://jsatjournal.org/`

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Evelyn Lamb

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GEOMETRY

Computer Search Settles 90-Year-Old Math Problem

10

By translating Keller's conjecture into a computer-friendly search for a type of graph, researchers have finally resolved a problem about covering spaces with tiles.

SAT Solvers

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from pycosat import solve

clauses = [ [-1, -2, -3], [1, -2], [2, -3], [3,
-1], [1, 2, 3] ]

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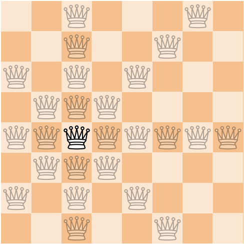
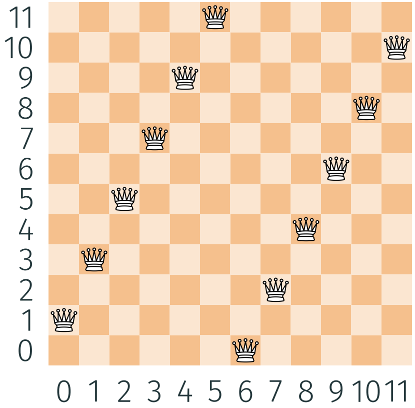
```
UNSAT
[1, 2, 3]
```

N Queens

Is it possible to place n queens on an $n \times n$ board such that no two of them attack each other?



Examples



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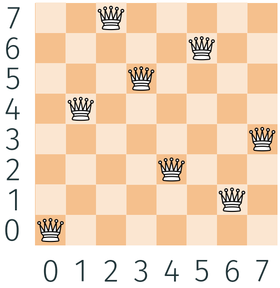

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- For $0 \le i < n$, ith row contains ≥ 1 queen:

$$(x_{i0} = 1 \text{ or } x_{i2} = 1 \text{ or } \dots \text{ or } x_{i(n-1)} = 1) .$$

Encoding as SAT

- n^2 0/1-variables: for $0 \le i, j < n$, $x_{ij} = 1$ iff queen is placed into cell (i, j)
- For $0 \le i < n$, ith row contains ≥ 1 queen:
$$(x_{i0} = 1 \text{ or } x_{i2} = 1 \text{ or } \dots \text{ or } x_{i(n-1)} = 1)\,.$$
- For $0 \le i < n$, ith row contains ≤ 1 queen:
$$\forall 0 \le j_1 \ne j_2 < n\colon\ (x_{ij_1} = 0 \text{ or } x_{ij_2} = 0)\,.$$

Encoding as SAT

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- For $0 \le i < n$, ith row contains ≥ 1 queen:
$$(x_{i0} = 1 \text{ or } x_{i2} = 1 \text{ or } \dots \text{ or } x_{i(n-1)} = 1)\,.$$
- For $0 \le i < n$, ith row contains ≤ 1 queen:
$$\forall 0 \le j_1 \neq j_2 < n\colon\ (x_{ij_1} = 0 \text{ or } x_{ij_2} = 0)\,.$$
- For $0 \le j < n$, jth column contains ≤ 1 queen:
$$\forall 0 \le i_1 \neq i_2 < n\colon\ (x_{i_1 j} = 0 \text{ or } x_{i_2 j} = 0)\,.$$

Encoding as SAT

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- For $0 \leq i < n$, ith row contains ≤ 1 queen:
$$\forall 0 \leq j_1 \neq j_2 < n\colon\; (x_{ij_1} = 0 \text{ or } x_{ij_2} = 0)\,.$$
- For $0 \leq j < n$, jth column contains ≤ 1 queen:
$$\forall 0 \leq i_1 \neq i_2 < n\colon\; (x_{i_1 j} = 0 \text{ or } x_{i_2 j} = 0)\,.$$
- For each pair $(i_1, j_1), (i_2, j_2)$ on diagonal:
$$(x_{i_1 j_1} = 0 \text{ or } x_{i_2 j_2} = 0)\,.$$

Implementation

```
from itertools import combinations, product
from pycosat import solve

n = 10
clauses = []


# converts a pair of integers into a unique integer
def varnum(i, j):
    assert i in range(n) and j in range(n)
    return i * n + j + 1


# each row contains at least one queen
for i in range(n):
    clauses.append([varnum(i, j) for j in range(n)])

# each row contains at most one queen
for i in range(n):
    for j1, j2 in combinations(range(n), 2):
        clauses.append([-varnum(i, j1), -varnum(i, j2)])

# each column contains at most one queen
for j in range(n):
    for i1, i2 in combinations(range(n), 2):
        clauses.append([-varnum(i1, j), -varnum(i2, j)])

# no two queens stay on the same diagonal
for i1, j1, i2, j2 in product(range(n), repeat=4):
    if i1 == i2:
        continue

    if abs(i1 - i2) == abs(j1 - j2):
        clauses.append([-varnum(i1, j1),
                        -varnum(i2, j2)])


assignment = solve(clauses)
for i, j in product(range(n), repeat=2):
    if assignment[varnum(i, j) - 1] > 0:
        print(j, end=' ')
```