Linear Programming

- Optimization problems: among all solutions satisfying certain constraints find optimal one.
LINEAR PROGRAMMING

• Optimization problems: among all solutions satisfying certain constraints find optimal one

• Find shortest cycle through all vertices
Linear Programming

• Optimization problems: among all solutions satisfying certain constraints find optimal one
• Find shortest cycle through all vertices
• Find optimal coloring
Linear Programming

- Optimization problems: among all solutions satisfying certain constraints find optimal one
- Find shortest cycle through all vertices
- Find optimal coloring
- Find minimum vertex cover
LINEAR PROGRAMMING

- Optimization problems: among all solutions satisfying certain constraints find **optimal** one
- Find **shortest cycle** through all vertices
- Find **optimal coloring**
- Find **minimum vertex cover**
- Linear programming: class of optimization problems where **constraints** and **optimization criterion** are linear functions
Avoiding Scurvy
• Orange costs $1, grapefruit costs $1; we have budget of $2/day
• Orange costs $1, grapefruit costs $1; we have budget of $2/day

• Orange weighs 100gm, grapefruit weighs 200gm, we can carry 300gm
• Orange costs $1, grapefruit costs $1; we have budget of $2/day

• Orange weighs 100gm, grapefruit weighs 200gm, we can carry 300gm

• Orange has 100gm of vitamin C, grapefruit has 150gm of vitamin C, maximize daily vitamin C intake.
Avoiding Scurvy. Plot

\[ \text{max } 2x + 3y \]

\[ x + y \leq 2 \]
\[ x + 2y \leq 3 \]
\[ x \geq 0 \]
\[ y \geq 0 \]
AVOIDING SCURVY. PLOT

\[
\text{max } 2x + 3y \\
x + y \leq 2 \\
x + 2y \leq 3 \\
x \geq 0 \\
y \geq 0
\]
max $2x + 3y$

$x + y \leq 2$

$x + 2y \leq 3$

$x \geq 0$

$y \geq 0$
\[
\begin{align*}
\text{max } & 2x + 3y \\
\text{s.t. } & x + y \leq 2 \\
& x + 2y \leq 3 \\
& x \geq 0 \\
& y \geq 0
\end{align*}
\]
Avoiding Scurvy. Plot

\[
\begin{align*}
\text{max } 2x + 3y \\
x + y &\leq 2 \\
x + 2y &\leq 3 \\
x &\geq 0 \\
y &\geq 0
\end{align*}
\]
**AVOIDING SCURVY. PLOT**

\[
\begin{align*}
\text{max } & 2x + 3y \\
\text{subject to } & x + y \leq 2 \\
& x + 2y \leq 3 \\
& x \geq 0 \\
& y \geq 0
\end{align*}
\]
AVOIDING SCURVY. PLOT

\[ \text{max } 2x + 3y \]

\[ x + y \leq 2 \]
\[ x + 2y \leq 3 \]
\[ x \geq 0 \]
\[ y \geq 0 \]
AVOIDING SCURVY. PLOT

\[
\begin{align*}
\max & \quad 2x + 3y \\
x + y & \leq 2 \\
x + 2y & \leq 3 \\
x \geq 0 \\
y \geq 0
\end{align*}
\]
Avoiding Scurvy. Plot

\[
\begin{align*}
\text{max } & 2x + 3y \\
x + y & \leq 2 \\
x + 2y & \leq 3 \\
x & \geq 0 \\
y & \geq 0 \\
\end{align*}
\]
AVOIDING SCURVY. PLOT

$max \ 2x + 3y$

$x + y \leq 2$

$x + 2y \leq 3$

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AVOIDING SCURVY. PLOT

\[
\begin{align*}
\max 2x + 3y \\
x + y &\leq 2 \\
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Profit Maximization
PROFIT MAXIMIZATION

• We have 6 machines and 20 workers

• A machine takes two workers to operate
• Each machine produces 20 chocolates/hour, each worker produces 5 chocolates/hour
• We need to produce at most 100 chocolates/hour
• Each chocolate costs $10, each worker gets $40 per hour
PROFIT MAXIMIZATION

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WORKERS AND MACHINES
Linear Classifier
Linear Classifier

• Given $n_1$ spam emails, and $n_2$ ham emails as points in $\mathbb{R}^d$
LINEAR CLASSIFIER

• Given $n_1$ spam emails, and $n_2$ ham emails as points in $\mathbb{R}^d$

• Find a linear function $h(a_1, \ldots, a_d)$ s.t.
• Given $n_1$ spam emails, and $n_2$ ham emails as points in $\mathbb{R}^d$

• Find a linear function $h(a_1, \ldots, a_d)$ s.t.
  • $h(a_1, \ldots, a_d) < 0$ for all spam emails
  • $h(a_1, \ldots, a_d) > 0$ for all ham emails
Linear Programming
LINEAR PROGRAMMING

• Find real numbers $x_1, \ldots, x_n$ that satisfy linear constraints

\[ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1 \]
\[ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geq b_2 \]
\[ \quad \vdots \]
\[ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \geq b_m \]
LINEAR PROGRAMMING

• Find real numbers $x_1, \ldots, x_n$ that satisfy linear constraints

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & \geq b_1 \\
    a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & \geq b_2 \\
    \vdots & \\
    a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & \geq b_m
\end{align*}
\]

• So that linear objective is \textit{maximized}

\[
    c_1x_1 + c_2x_2 + \ldots + c_nx_n
\]
• Turn *minimization* problem into *maximization* problem:

\[
\begin{align*}
\text{min} & \quad c_1 x_1 + c_2 x_2 + \ldots - c_n x_n \\
\text{max} & \quad -c_1 x_1 - c_2 x_2 - \ldots - c_n x_n
\end{align*}
\]
EQUIVALENT FORMULATIONS

• Turn minimization problem into maximization problem:

\[
\begin{align*}
\text{min} & \quad c_1x_1 + c_2x_2 + \ldots - c_nx_n \\
\text{max} & \quad -c_1x_1 - c_2x_2 - \ldots - c_nx_n
\end{align*}
\]

• Turn \( \leq \) into \( \geq \):

\[
\begin{align*}
& a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \\
& -a_{11}x_1 - a_{12}x_2 - \ldots - a_{1n}x_n \geq -b_1
\end{align*}
\]
• Turn $=$ into $\geq$:

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
  a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\geq b_1 \\
  -a_{11}x_1 - a_{12}x_2 - \ldots - a_{1n}x_n &\geq -b_1
\end{align*}
\]
Matrix Formulation

Input is a matrix $A \in \mathbb{R}^{m \times n}$, and vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$
**MATRIX FORMULATION**

Input is a matrix $A \in \mathbb{R}^{m \times n}$, and vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$

$$Ax = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 & \ldots & a_{1n}x_n \\ \vdots & \ddots & \vdots \\ a_{m1}x_1 & \ldots & a_{mn}x_n \end{bmatrix} \geq \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$
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$Ax \geq b$
Matrix Formulation

Input is a matrix \( A \in \mathbb{R}^{m\times n} \), and vectors \( b \in \mathbb{R}^m \) and \( c \in \mathbb{R}^n \)

\[
Ax = \begin{bmatrix}
a_{11} & \ldots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{m1} & \ldots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}
= \begin{bmatrix}
a_{11}x_1 & \ldots & a_{1n}x_n \\
\vdots & \ddots & \vdots \\
a_{m1}x_1 & \ldots & a_{mn}x_n
\end{bmatrix}
\geq \begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_m
\end{bmatrix}
\]

\[Ax \geq b\]

maximize \( cx = \begin{bmatrix} c_1 & \ldots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\
\vdots \\
x_n \end{bmatrix} = c_1x_1 + \ldots c_nx_n \]
HISTORY OF LINEAR PROGRAMMING

• Kantorovich, 1939, started studying Linear Programming
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• Kantorovich, 1939, started studying Linear Programming
• Dantzig, 1947, developed **Simplex Method** for US Air force planning problems
• Koopmans, 1947, showed how to use LP for analysis of economic theories
• Kantorovich and Koopmans won Nobel Prize in Economics in 1971
• Dantzig's algorithm is "One of top 10 algorithms of the 20th century"
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**Theorem**

A linear function takes its maximum and minimum values on vertices.
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- Start at any vertex
Theorem

A linear function takes its maximum and minimum values on vertices

- Start at any vertex
- While there is an adjacent vertex with higher profit
  - Move to that vertex
CORNER CASES

- No solutions
CORNER CASES

• No solutions

• Unbounded profit
ALGORITHMS FOR LINEAR PROGRAMMING

- Simplex method
ALGORITHMS FOR LINEAR PROGRAMMING

• Simplex method

• Many professional packages that implement efficient algorithms for LP
ALGORITHMS FOR LINEAR PROGRAMMING

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• Last week!
Ellipsoid Method