GEMS OF TCS

GÖDEL’S INCOMPLETENESS

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Gödel’s Incompleteness Theorem
Axiomatization of Math

• Find a set of simple and obvious axioms
Axiomatization of Math

- Find a set of simple and obvious axioms
- Any proof could be (in principle) traced back to this set of axioms
EUCLID’S AXIOMS

• For any pair of distinct points, there is exactly one line connecting them
Euclid’s Axioms

• For any pair of distinct points, there is exactly one line connecting them
• Any line segment can be extended to an infinite line
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• Any line segment can be extended to an infinite line
• For any pair of distinct points, there is exactly one circle centered at the first and touching the second
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• All right angles are equal to one another
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• [The Parallel Postulate] Given a line $L$ and a point $x$, there is exactly one line parallel to $L$ that passes through $x$
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PEANO ARITHMETIC

• 0 is a natural number
Peano Arithmetic

- 0 is a natural number
- \( \forall x, x = x \)
- If \( x = y \), then \( y = x \)
- If \( x = y \) and \( y = z \), then \( x = z \)
- ...
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- If \( x = y \), then \( y = x \)
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- ... 
- \( \forall x, y, x = y \text{ iff } \text{Next}(x) = \text{Next}(y) \)
- If \( x \) is a natural number, then \( \text{Next}(x) \) is a natural number
- ...
Peano Arithmetic

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- If x = y, then y = x
- If x = y and y = z, then x = z
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- ∀x, y, x = y iff Next(x) = Next(y)
- If x is a natural number, then Next(x) is a natural number
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- ∀x, y, x + Next(y) = Next(x + y)
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- \( \forall x, y, x \cdot \text{Next}(y) = x \cdot y + x \)
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• ∀x, y, x + Next(y) = Next(x + y)
• ∀x, y, x · Next(y) = x · y + x
• Induction
Naive Set Theory

- Set
- Membership in a Set
- Empty Set
- Equality
RUSSELL’S PARADOX

The barber is the “one who shaves all those, and those only, who do not shave themselves”. The question is, does the barber shave himself?
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PRINCIPIA MATHEMATICA
ZFC
Gödel’s Incompleteness Theorem

Any attempt to axiomatize all of mathematics is guaranteed to fail
• Function HALT is defined as follows.
HALTING PROBLEM

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  • The first input is algorithm A
  • HALT(A, x) = 1 if A halts on input x
  • HALT(A, x) = 0 if A enters infinite loop on input x

HALT is undecidable (Lecture 13)
Function HALT is defined as follows.

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- HALT is undecidable (Lecture 13).