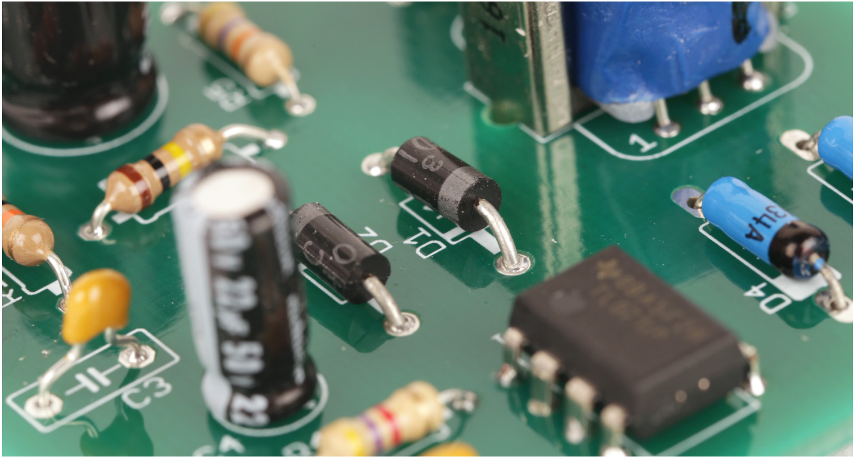
Gems of TCS

Circuit Complexity

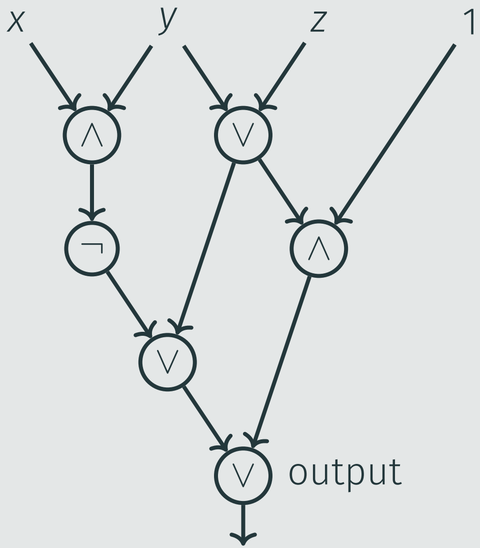
Sasha Golovnev

October 19, 2022



C3
D2
D1
D4

Circuit



Definition

A circuit is a directed acyclic graph of in-degree at most 2. Nodes of in-degree 0 are called inputs and are marked by Boolean variables and constants. Nodes of in-degree 1 and 2 are called gates: gates of in-degree 1 are labeled with NOT, gates of in-degree 2 are labeled with AND or OR. One of the sinks is marked as output.

Boolean Circuits

$$f\colon \{0,1\}^n \to \{0,1\}$$

$$\begin{aligned}
g_1 &= \neg x_1 \\
g_2 &= x_2 \wedge x_3 \\
g_3 &= g_1 \vee g_2 \\
g_4 &= g_2 \vee 1 \\
g_5 &= g_3 \wedge g_4
\end{aligned}$$

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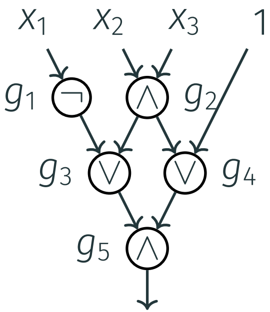

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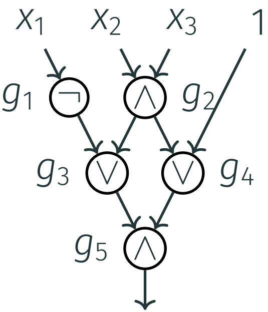

Inputs:
$x_1, \ldots, x_n, 0, 1$

Gates:
AND, OR, NOT

Fan-out:
unbounded

Depth:
unbounded

Exponential Bounds

Lower Bound [Sha1949]

Almost all functions of n variables have circuit size

$$\geq 2^n/n$$

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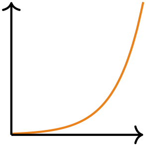
Upper Bound [Lup1958]

Any function can be computed by a circuit of size

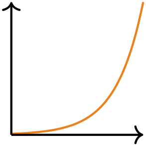
$$\leq 2^n/n$$

Explicit Bounds



Most functions have exponential circuit complexity

Explicit Bounds



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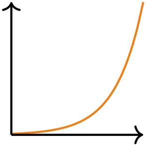
$$P \neq NP$$

We want to prove super-polynomial lower bounds

Explicit Bounds



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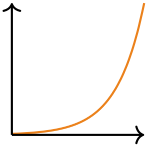

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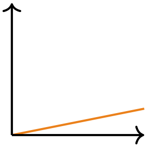
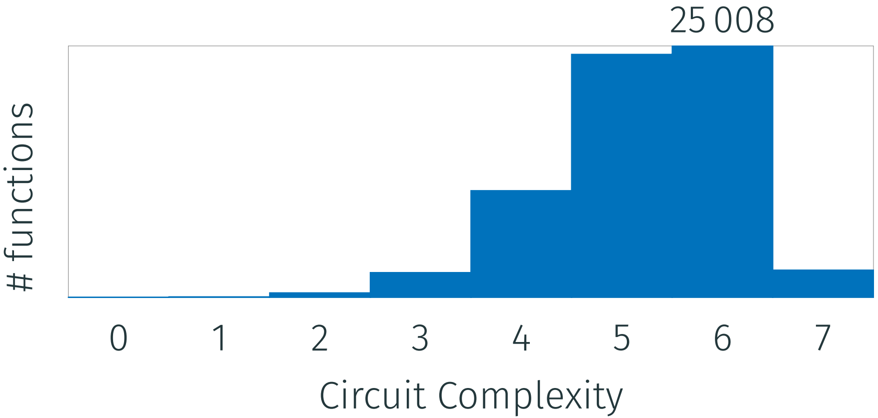


We can prove only $\approx 5n$ lower bounds

Circuit Complexity: $n = 4$



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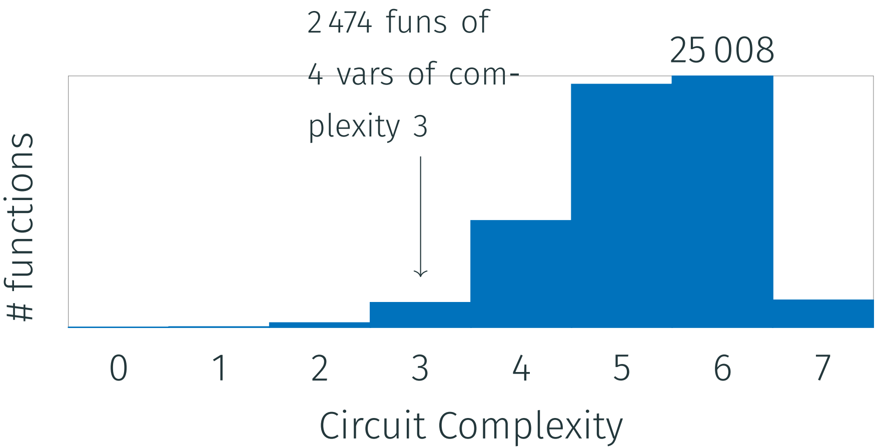
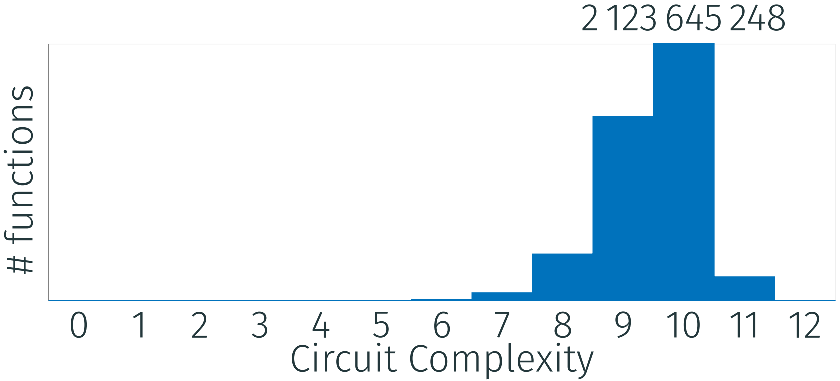
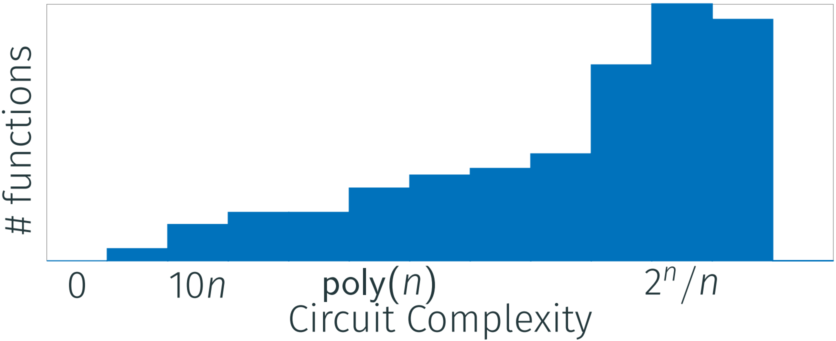


Circuit Complexity: $n = 5$



Circuit Complexity: General n



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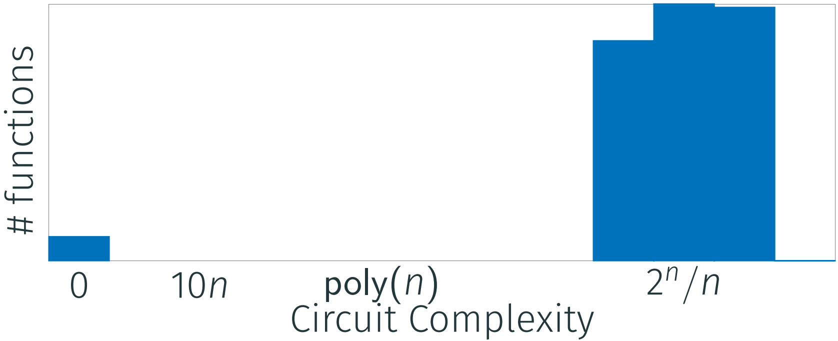

Hierarchy Theorem

Theorem

For any $T \leq 2^n/n$, there is a function $f\colon \{0,1\}^n \to \{0,1\}$ s.t.

$$\mathsf{Size}(f) = T \pm n\,.$$

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$$\text{Size}(h) \geq 2^n/n$$

$$h\colon \{0,1\}^n \to \{0,1\}$$

$$y_1, \ldots, y_k \in \{0,1\}^n$$

$$h(y_i) = 1$$

Hybrid Method

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$$\text{Size}(g_{i+1}) \leq \text{Size}(g_i) + 2n$$

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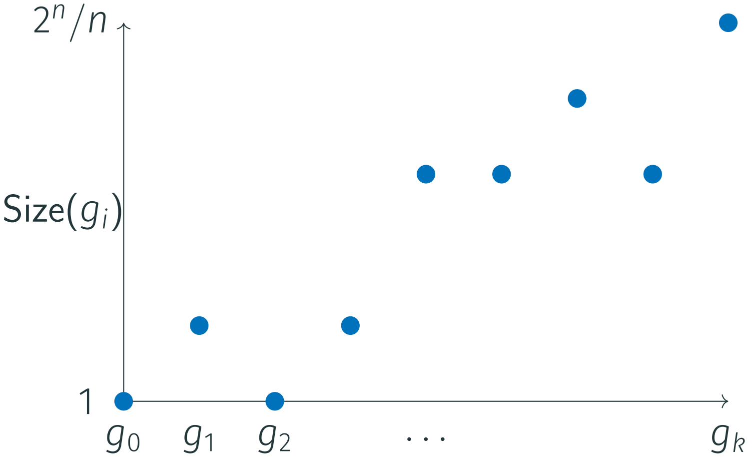

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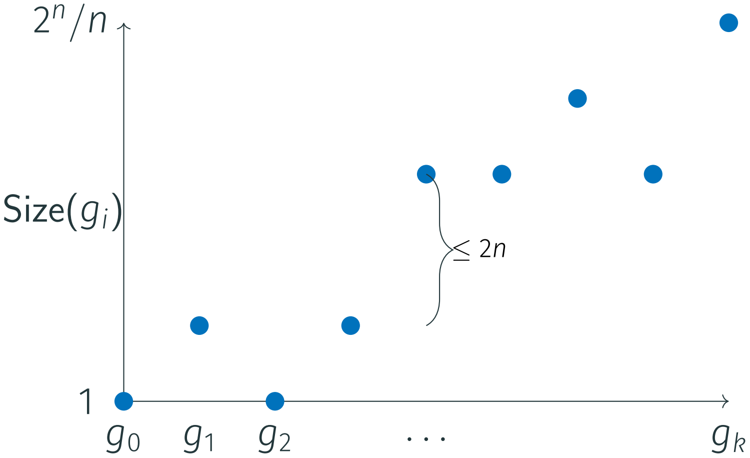

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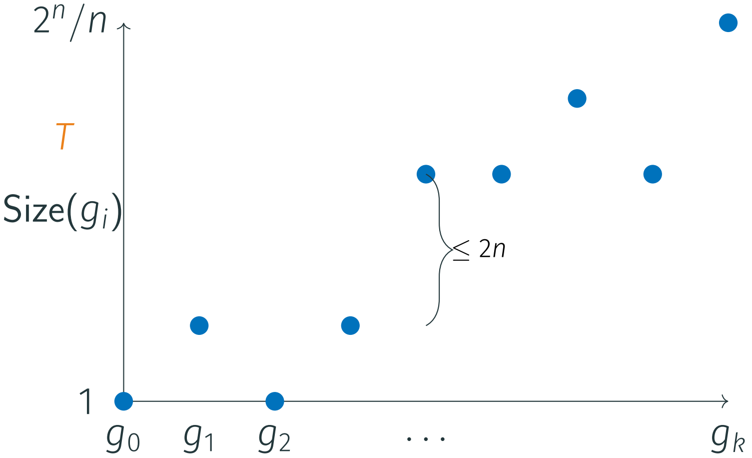

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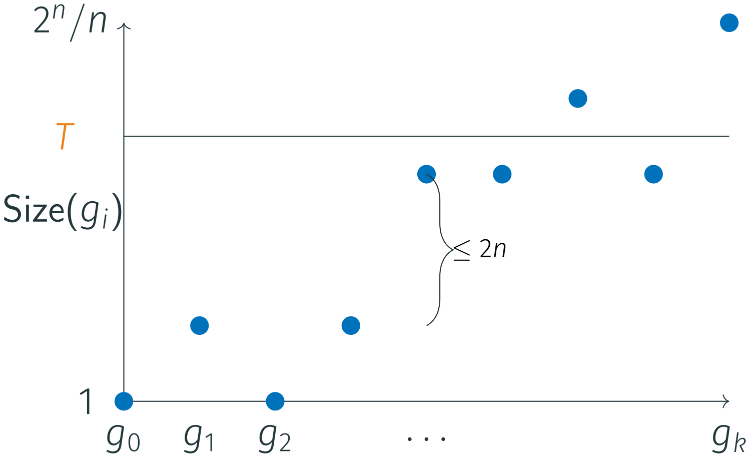

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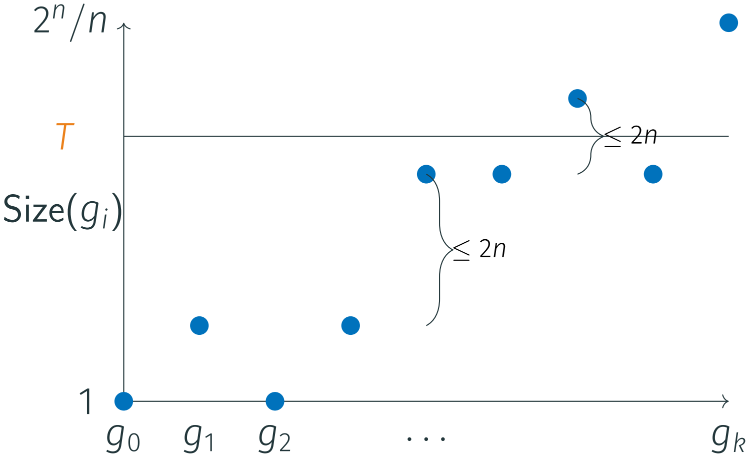

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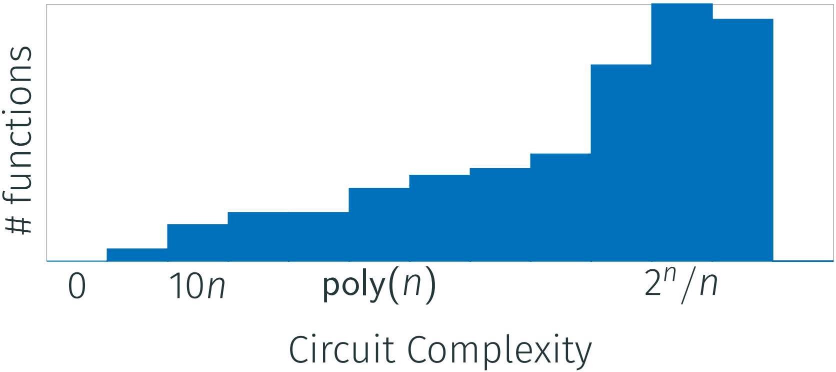

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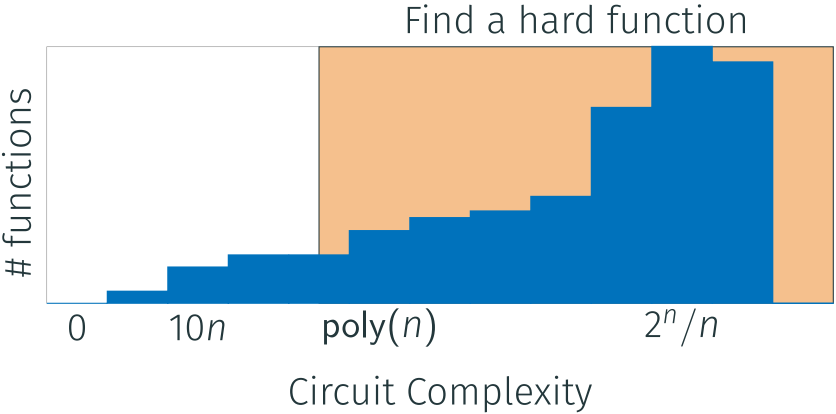

Find a hard function
functions
0
10n
poly(n)
$2^n/n$
Circuit Complexity

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- Goal: Find a hard function

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- Lower bounds: what functions are hard

Circuit Complexity

- Goal: Find a hard function
- Lower bounds: what functions are hard
- Upper bounds: what functions are easy

Circuit Upper Bound. Proof

Upper Bound [Lup1958]

Any function can be computed by a circuit of size

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$$f(x_1, \dots, x_n) = \begin{cases} f(1, x_2, \dots, x_n), & \text{if } x_1 = 1 \\ f(0, x_2, \dots, x_n), & \text{if } x_1 = 0 \end{cases}$$

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$$= (x_1 \wedge f(1, x_2, \ldots, x_n)) \vee (\bar{x}_1 \wedge f(0, x_2, \ldots, x_n))$$

$$= (x_1 \wedge g_1(x_2, \ldots, x_n)) \vee (\bar{x}_1 \wedge g_0(x_2, \ldots, x_n))$$

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$$\mathsf{Size}(n) \leq 4 + 2\,\mathsf{Size}(n-1) = O(2^n)$$

Circuit Lower Bound. Proof

Lower Bound [Sha1949]

Almost all functions of n variables have circuit size

$$\geq 2^n/(10n)$$