Gems of TCS

Approximation Algorithms

Sasha Golovnev

August 31, 2021

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- When do we use approximation algorithms?

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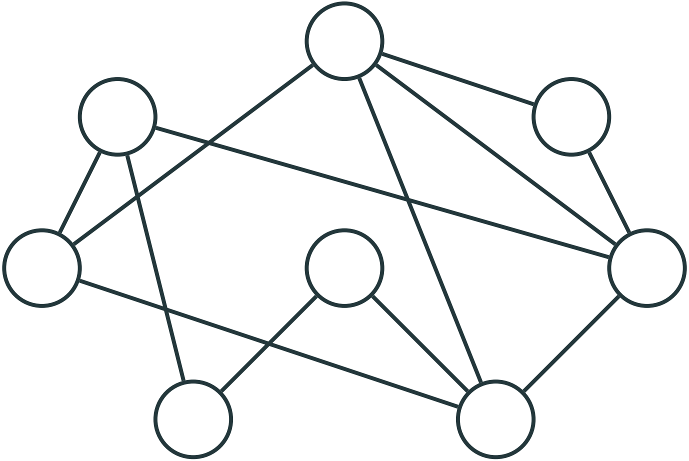
- A Maximum Matching is a matching of the largest size

Matchings. Examples



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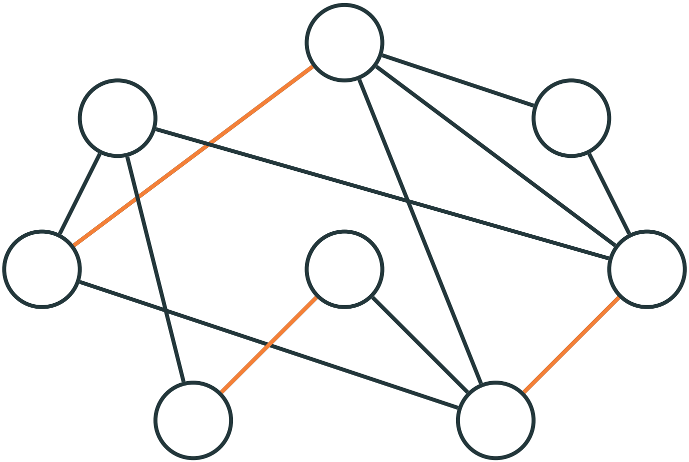

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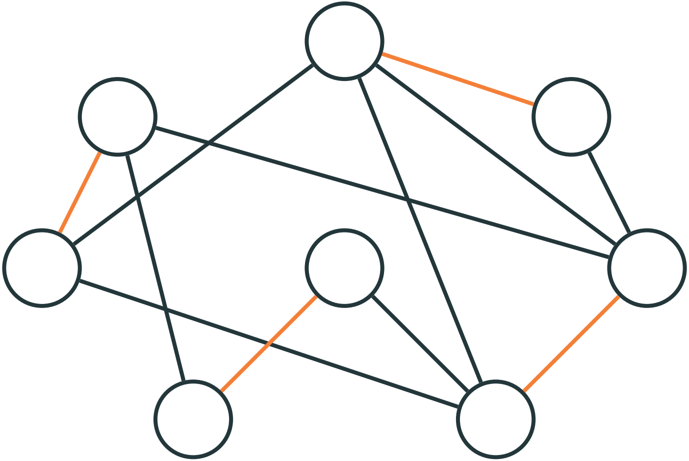
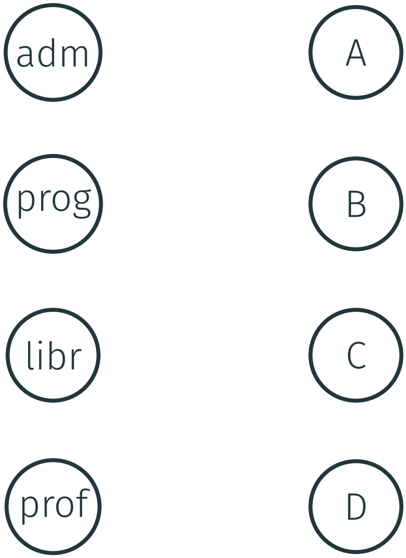


Job Assignment

	Alice	Ben	Chris	Diana
Administrator	+		+	
Programmer		+	+	
Librarian	+	+		
Professor				+

Job Assignment



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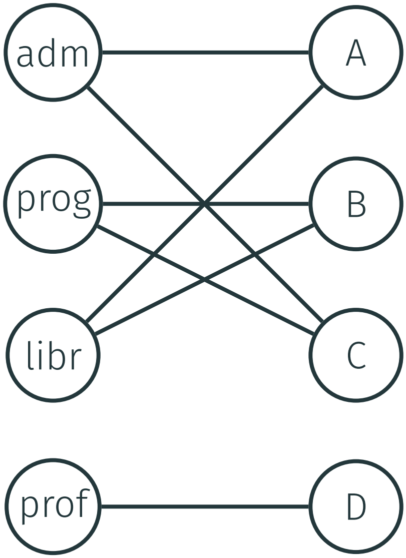

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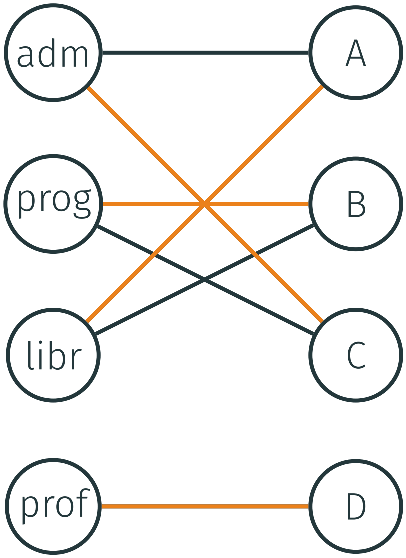
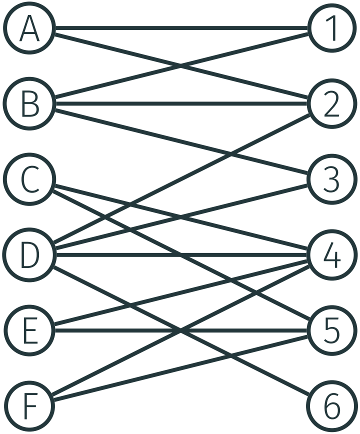


Room Assignment

	R# 1	R# 2	R# 3	R# 4	R# 5	R# 6
Aaron	+	+				
Bianca	+	+	+			
Carol				+	+	
Dana		+	+	+		+
Emma				+	+	
Francis				+	+	

Room Assignment



Algorithms

Maximal Matching

Can be found in polynomial time by a greedy algorithm

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Can be found in polynomial time by the blossom algorithm

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Minimum Weight Perfect Matching

Can be found in polynomial time by Edmonds' algorithm

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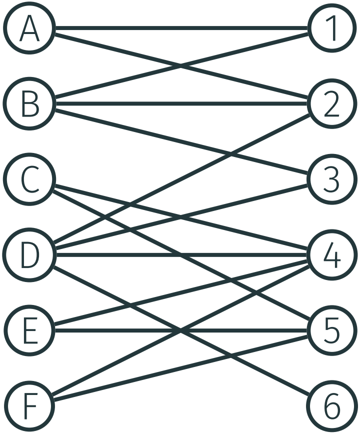

Vertex Cover

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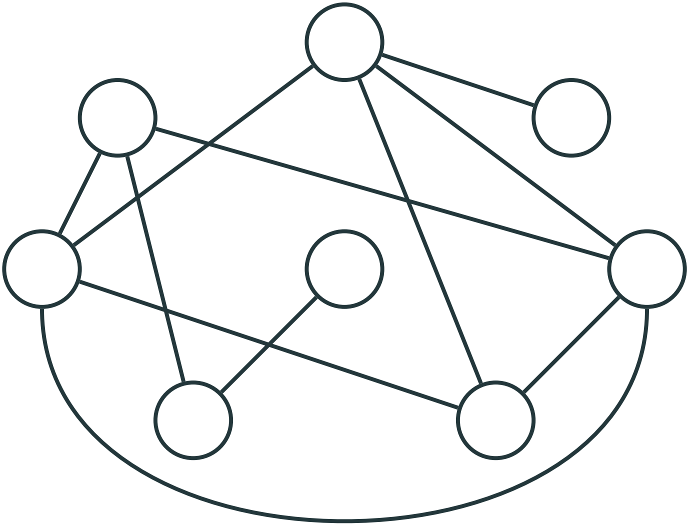
- A Minimum Vertex Cover is a vertex cover of the smallest size.

Vertex Covers: Examples



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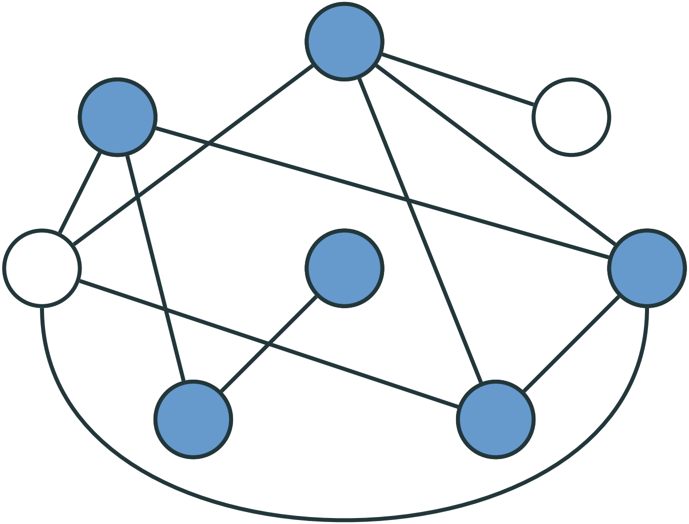

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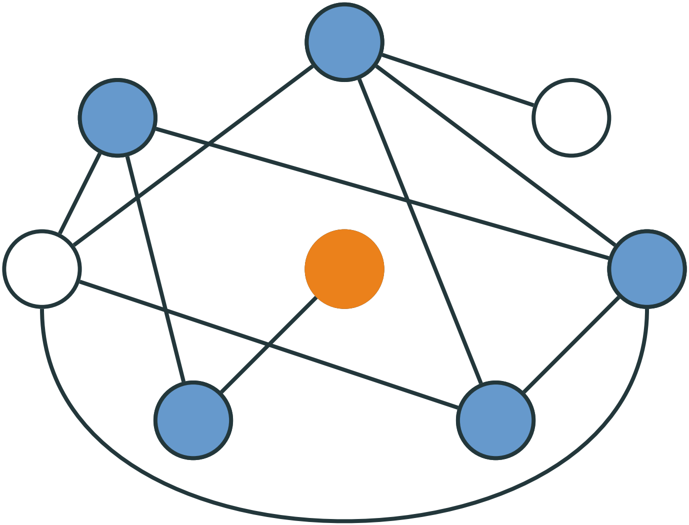

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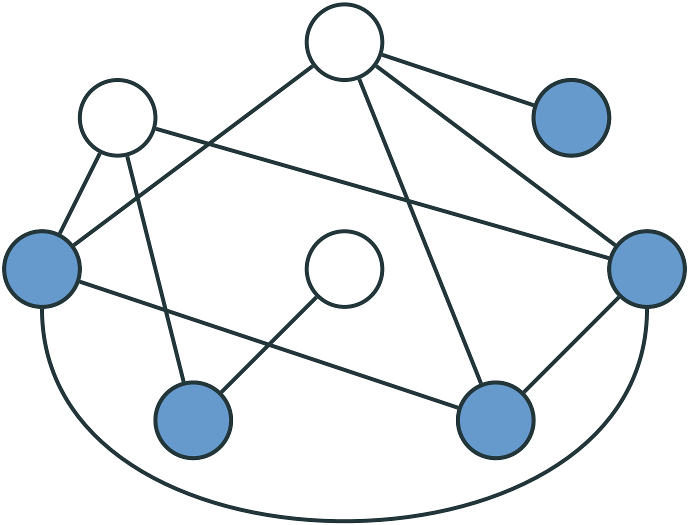

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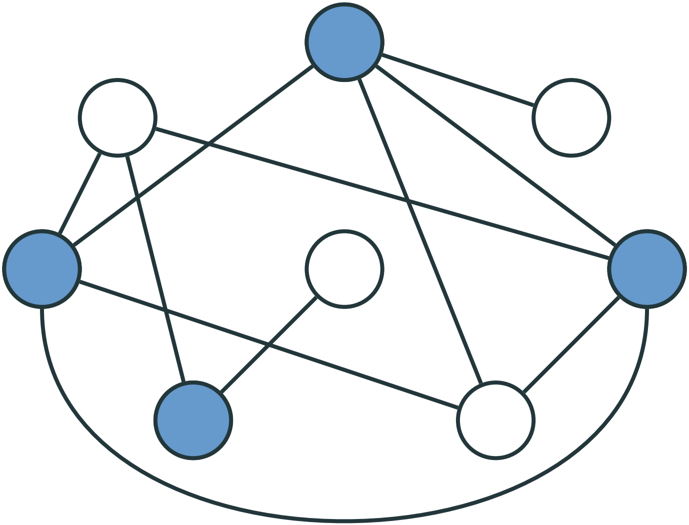
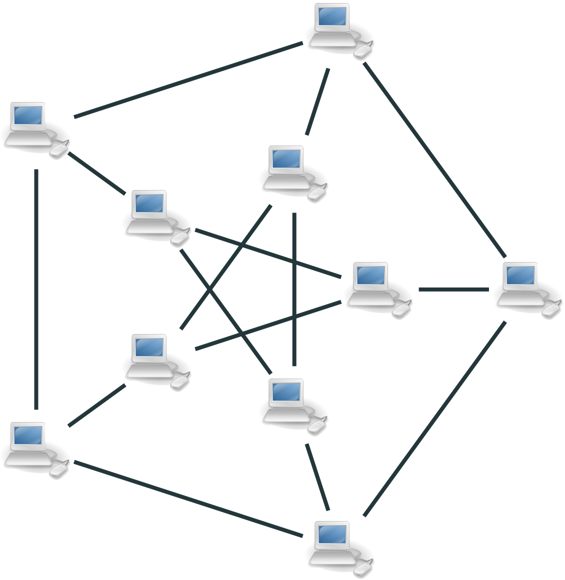


Antivirus System



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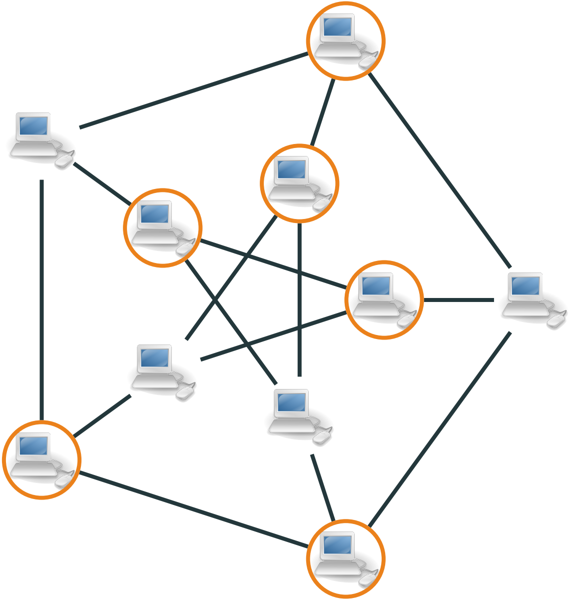

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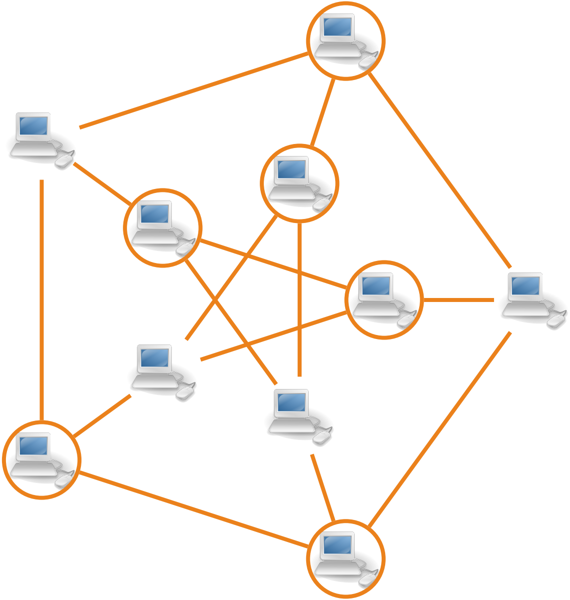

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Minimum Vertex Cover

Is **NP**-hard. We only know exponential-time algorithms

Approximation Algorithm

- $M \leftarrow$ maximal matching in G

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- return all vertices in M

Equivalent Algorithm

- $C \leftarrow \emptyset$

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- while $E \neq \emptyset$
  - $\{u, v\} \leftarrow$ any edge from E
  - add u, v to C
  - delete from E all edges incident to u or v
- return C

Proof

Lemma

This algorithm runs in polynomial time and is 2-approximate: it returns a vertex cover that is at most twice larger then a minimum vertex cover.

Final Remarks

- The analysis is tight: there are graphs with matchings twice larger than vertex covers

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- No 1.99-approximation algorithm is known

Break

Matchings:

http://bit.ly/job-assignment

Vertex covers:

http://bit.ly/antivirus-system

Traveling Salesman

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- If $\mathsf{P} \neq \mathsf{NP}$, then there is no k-approximation algorithm for the general version of TSP for any constant k
- Euclidean TSP: $w(u, v) = w(v, u)$ and $w(u, v) \leq w(u, z) + w(z, v)$
- We will design a 2-approximation algorithm: it quickly finds a cycle that is at most twice longer than an optimal one

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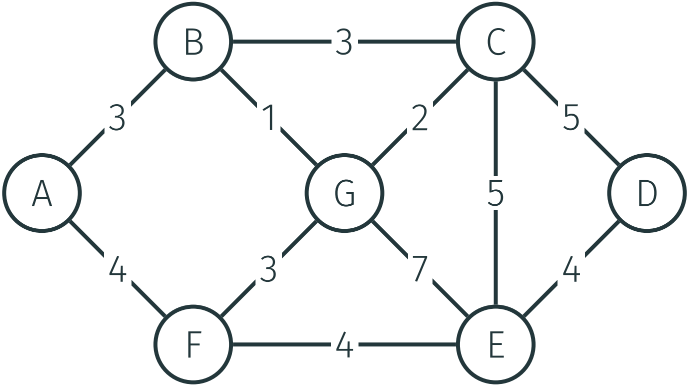
- A Spanning Tree of a graph G is a subgraph of G that (i) is a tree and (ii) contains all vertices of G
- A Minimum Spanning Tree of a weighted graph G is a spanning tree of the smallest weight

Minimum Spanning Tree: Examples



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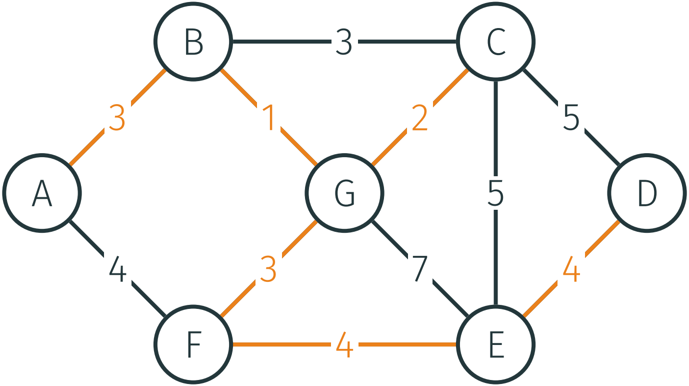

Minimum Spanning Trees

Lemma

Let G be an undirected graph with non-negative edge weights. Then $\mathsf{MST}(G) \leq \mathsf{TSP}(G)$.

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Proof

By removing any edge from an optimum TSP cycle one gets a spanning tree of G.

Eulerian Cycle

An Eulerian cycle (or path) visits every edge exactly once

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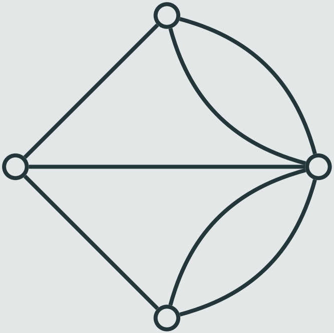
Criteria

A connected undirected graph contains an Eulerian cycle, if and only if the degree of every node is even

Example

Non-Eulerian graph



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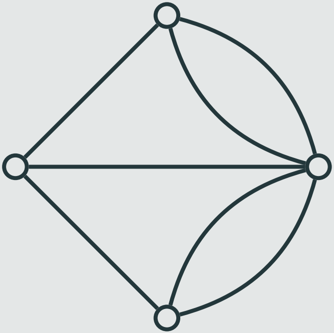
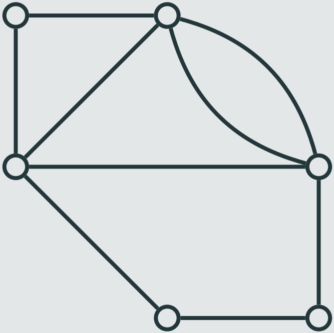


Eulerian graph



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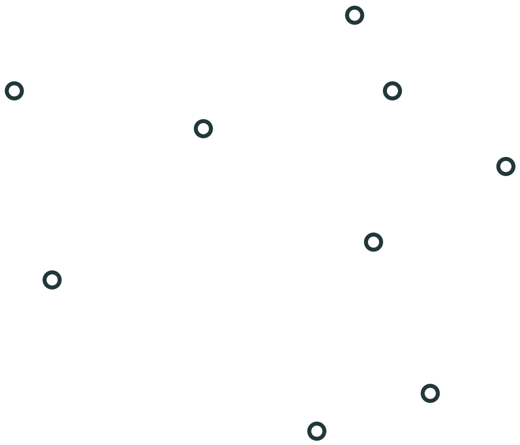

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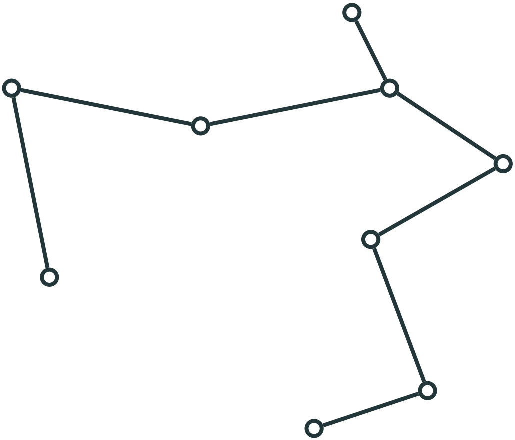

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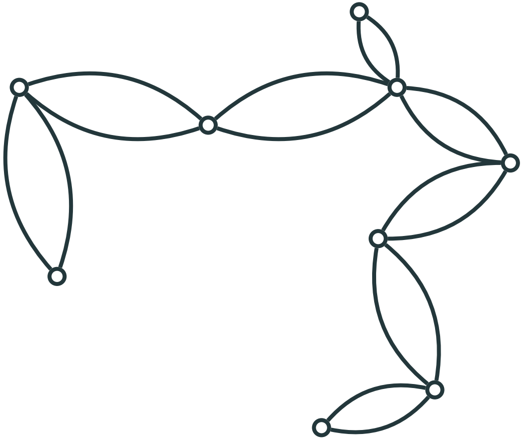

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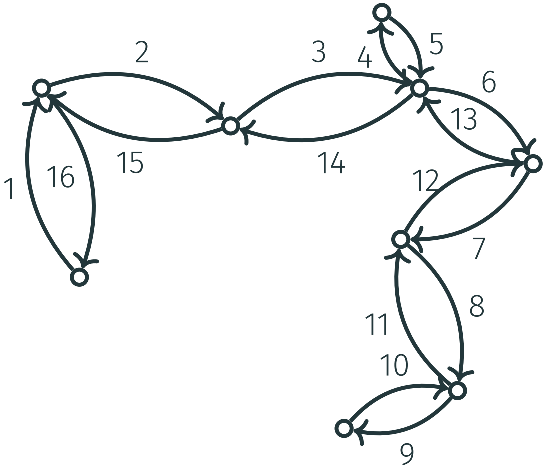

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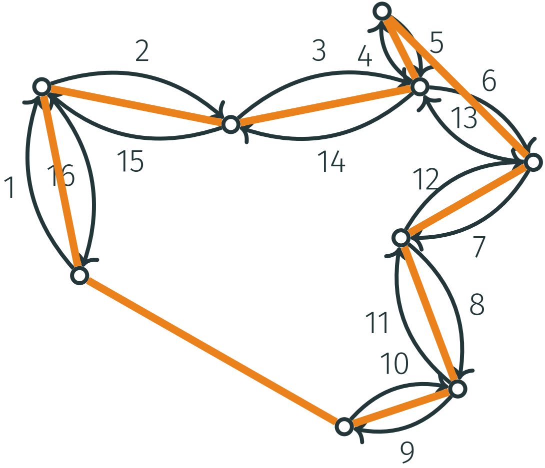

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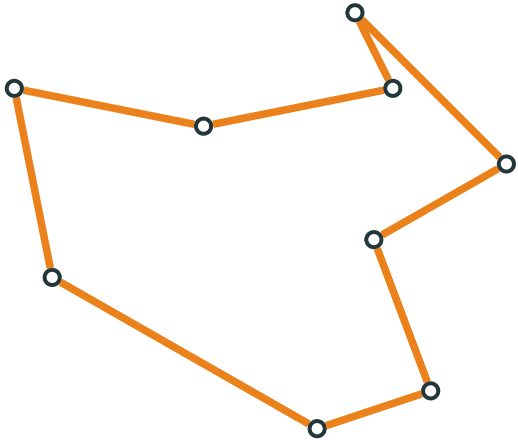

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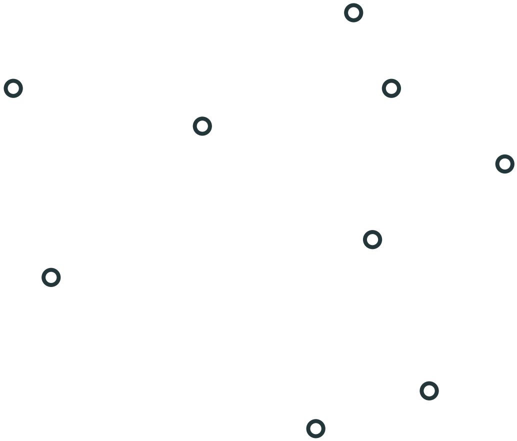
- The total length of the MST $T \leq \mathrm{OPT}$
- We start with Eulerian cycle of length $2|T|$
- Shortcuts can only decrease the total length

Improvement



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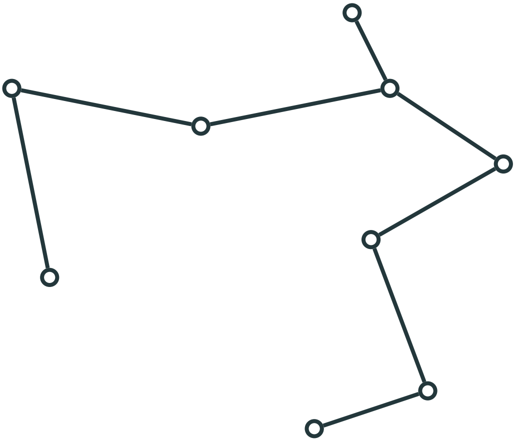

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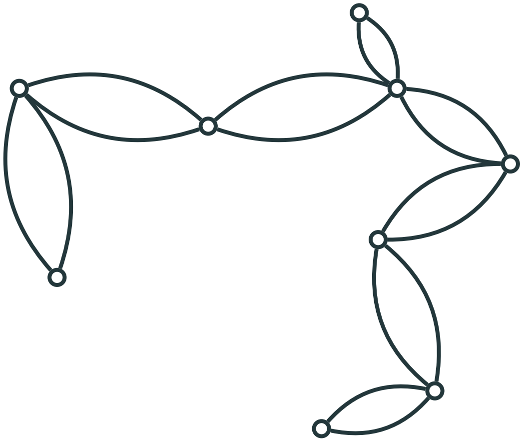

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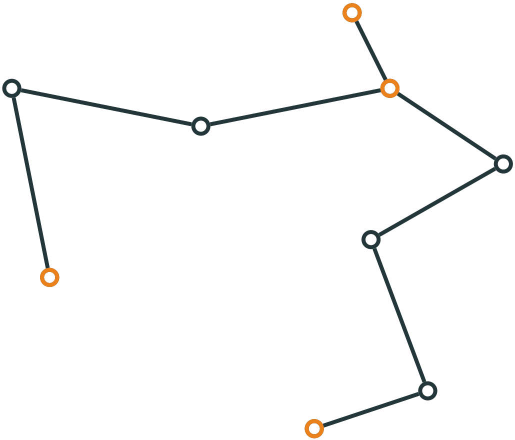

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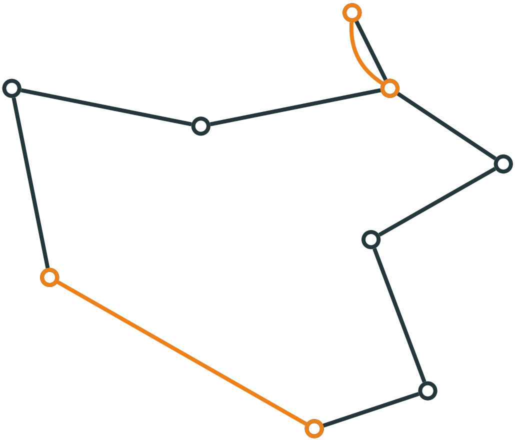

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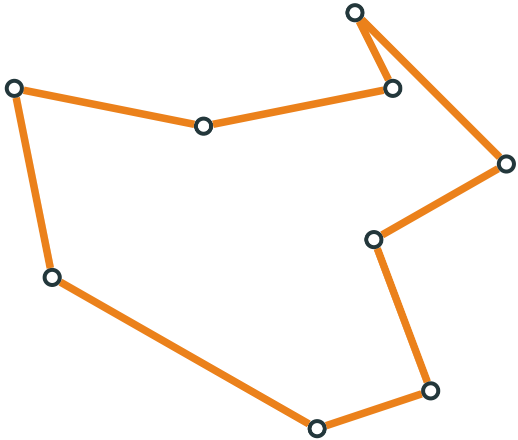

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Final Remarks

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- The currently best known approximation algorithm for TSP with triangle inequality is has approximation factor of $3/2 - 10^{-36}$ (July 2020)