Gems of TCS

Randomized Algorithms

Sasha Golovnev

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- $A_1 = \{HH\}$, $A_2 = \{HT\}$,
  $\Pr[A_1 \cup A_2] = \Pr[A_1] + \Pr[A_2]$

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- Throw two dice:
  $Y =$ sum of numbers, $Z = \max$ of numbers
- Expected value $\mathbb{E}[X] = \sum_i \Pr[x_i] \cdot x_i$
- Throw a die, $X =$ the number you're getting

$$\mathbb{E}[X] = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \ldots + \frac{1}{6} \cdot 6 = 3.5$$

Cloud Sync

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- Can send $n - 1$ bits?

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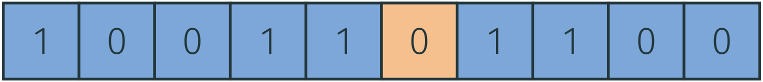

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Randomized algorithm can solve the problem by sending $\approx \log n$ bits!

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$a \in \{0, \ldots, 2^n - 1\}$

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$a \mod p$

EQ iff $a = b \mod p$

$b \in \{0, \ldots, 2^n - 1\}$

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- Prime Number Theorem: there are $\approx N / \log N$ prime numbers in the interval $\{2, 3, \dots, N\}$
- With probability $\approx 1 - \frac{1}{100n}$ the output is correct

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$\mathbb{E}[X + Y]$?

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- By linearity of expectation:

$$\begin{aligned} &\mathbb{E}[X_1 + X_2 + X_3 + X_4 + X_5] \\ &= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + \mathbb{E}[X_4] + \mathbb{E}[X_5] \\ &= 5 \cdot 3.5 = 17.5 \end{aligned}$$

Break

- Alice and Bob have (unusual) dice
- Numbers on Alice's die are 2, 2, 2, 2, 3, 3
- Numbers on Bob's die are 1, 1, 1, 1, 6, 6
- Alice and Bob throw their dice; the one with the larger number on the die wins
- Whose die has larger expected number?
- Who wins with higher probability?

Maximum Cut (Max-CUT)

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- Expected number of cut edges

$$\mathbb{E}[\sum_{(u,v)\in E} X_{u,v}] = \sum_{(u,v)\in E} \mathbb{E}[X_{u,v}] = |E|/2$$

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- $\mathbb{E}[\delta(S)] \geq \mathrm{OPT}/2$
- Can we have algorithm that always outputs $\delta(S) \geq \mathrm{OPT}/2$?

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$$\Pr[X \geq 5\mathbb{E}[X]] \leq \frac{1}{5}.$$

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- By our assumption at least $\frac{n}{100}$ tickets win at least 500 dollars

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- This exceeds the total prize budget of $4n$!
- Contradiction!

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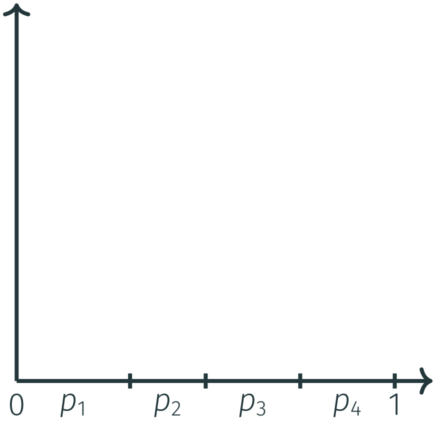

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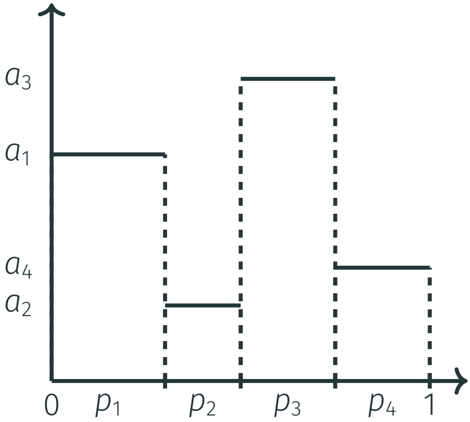

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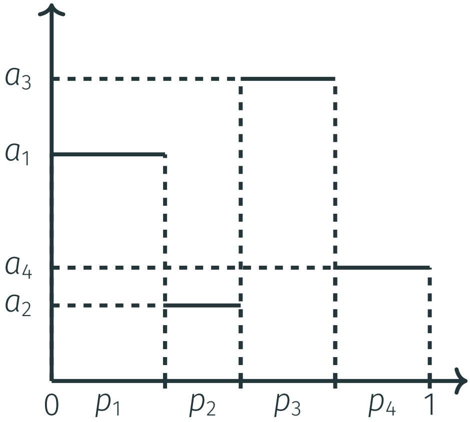

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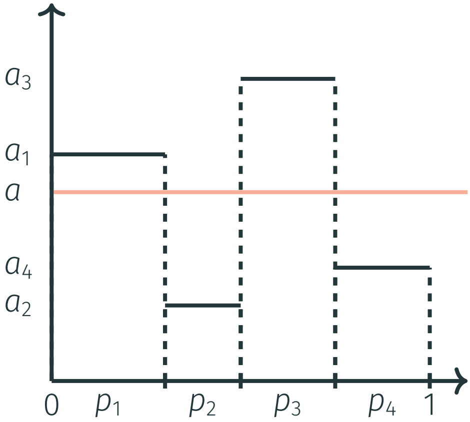

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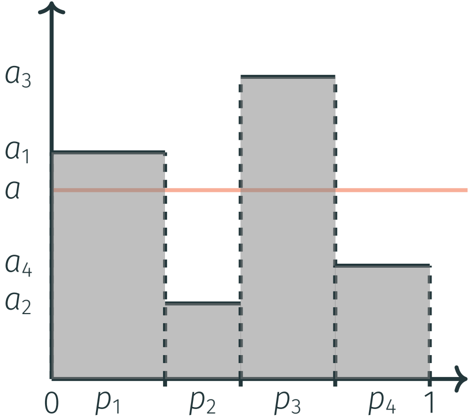

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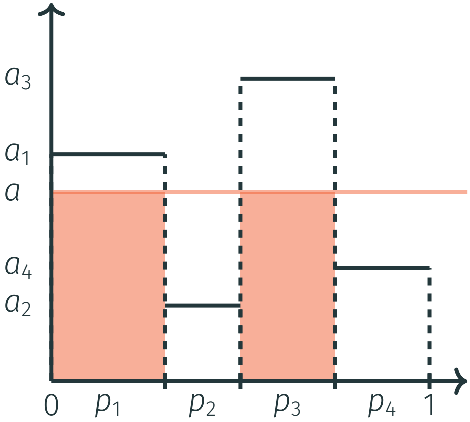

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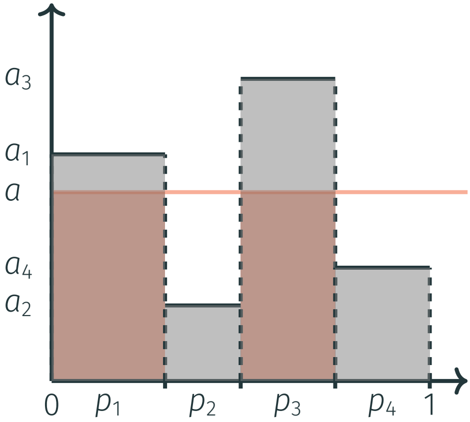

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The gray region is larger: the inequality follows

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- Ex. $\varepsilon = 1/100$: with probability at least $1/200$, we have 2.03-approximation

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- We have $\frac{2}{1-\varepsilon}$-approximation with probability $1 - \frac{1}{10^{10}n}$

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- We can amplify probability of success by independent repetitions