GEMS OF TCS

RANDOMIZED ALGORITHMS

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August 31, 2021
RANDOMIZED ALGORITHMS

• Randomized algorithm may be faster and simpler

• For some tasks randomness is necessary

• We'll use randomized algorithms in virtually all following topics

• Randomized algorithms make mistakes (with small probability)
Randomized Algorithms

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Randomized Algorithms

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- For some tasks randomness is necessary
- We’ll use randomized algorithms in virtually all following topics
- Randomized algorithms make mistakes (with small probability)
• Sample Space $\Omega$. 
• **Sample Space** $\Omega$.

$\Omega = \{1, 2, 3, 4, 5, 6\}$;
• Sample Space $\Omega$.

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• **Event** $A \subseteq \Omega$. 
• **Sample Space** $\Omega$.
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• **Event** $A \subseteq \Omega$. $A = \{2, 4, 6\}$;
Review of Probability Theory

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  • $\Pr(\Omega) = 1$
Review of Probability Theory

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- **Probability measure**: $\forall A, \Pr(A) \in [0, 1]$
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  - $A_1, A_2, \ldots$ are disjoint: $\Pr[\bigcup_i A_i] = \sum_i \Pr[A_i]$
Review of Probability Theory

- **Sample Space** \( \Omega \).
  \[\Omega = \{1, 2, 3, 4, 5, 6\}; \quad \Omega = \{HH, HT, TH, TT\}\]

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- **Probability measure**: \( \forall A, \Pr(A) \in [0, 1] \)
  - \( \Pr(\Omega) = 1 \)
  - \( A_1, A_2, \ldots \) are disjoint: \( \Pr[\bigcup_i A_i] = \sum_i \Pr[A_i] \)
  - \( A_1 = \{HH\}, \quad A_2 = \{HT\}, \quad \Pr[A_1 \cup A_2] = \Pr[A_1] + \Pr[A_2] \)
INDEPENDENT EVENTS

• $A_1$ and $A_2$ are independent iff

$$\Pr[A_1 \cap A_2] = \Pr[A_1] \cdot \Pr[A_2]$$
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\[ \Pr[A_1] = 1/6; \]
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\[ \Pr[A_1] = \frac{1}{6};\ \Pr[A_2] = \frac{1}{6};\ \Pr[A_1 \cap A_2] = \frac{1}{36} \]
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• $A_1 = \{\text{1st die is 6}\}$, $A_2 = \{\text{2nd die is 6}\}$

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Random Variable

- Result of experiment is often not event but number
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  $Y =$ sum of numbers, $Z =$ max of numbers
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  - $Y =$ sum of numbers, $Z =$ max of numbers
- Expected value $\mathbb{E}[X] = \sum_i \Pr[x_i] \cdot x_i$
**Random Variable**

- Result of experiment is often not event but number
- Random variable \( X: \Omega \rightarrow \mathbb{R} \)
- Toss three coins, \( X = \) number of heads
- Throw two dice:
  \( Y = \) sum of numbers, \( Z = \max \) of numbers
- Expected value \( \mathbb{E}[X] = \sum_i \Pr[x_i] \cdot x_i \)
- Throw a die, \( X = \) the number you’re getting

\[
\mathbb{E}[X] = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \ldots + \frac{1}{6} \cdot 6 = 3.5
\]
Cloud Sync
Cloud Sync

- Synchronize local files to the cloud
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- Has file been changed? File length: $n$ bits
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- Algorithm: send $n$ bits
**Cloud Sync**

- Synchronize local files to the cloud
- Has file been changed? File length: $n$ bits
- Algorithm: send $n$ bits
- Can send $n - 1$ bits?
No algorithm can solve the problem by sending $n - 1$ bits. Randomized algorithm can solve the problem by sending $\approx \log n$ bits!
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RANDOMIZED ALGORITHM

local file

1 0 0 1 1 0 1 1 0 0

cloud file

1 0 0 1 1 1 1 1 1 1 0 0
**RANDOMIZED ALGORITHM**

**local file**

```
1 0 0 1 1 0 1 1 0 0
```

\[ a \in \{0, \ldots, 2^n - 1\} \]

**cloud file**

```
1 0 0 1 1 1 1 1 1 0 0
```
RANDOMIZED ALGORITHM

local file

| 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 |

\( a \in \{0, \ldots, 2^n - 1\} \)

cloud file

| 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |

\( b \in \{0, \ldots, 2^n - 1\} \)
## Randomized Algorithm

Local file

\[
a \in \{0, \ldots, 2^n - 1\}
\]

Pick random prime \( p \in \{2, 3, \ldots, 100n^2 \log n\} \)

Cloud file

\[
b \in \{0, \ldots, 2^n - 1\}
\]
**Randomized Algorithm**

**local file**

| 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 |

\[ a \in \{0, \ldots, 2^n - 1\} \]

\[ a \mod p \]

Pick random prime \( p \in \{2, 3, \ldots, 100n^2 \log n\} \)

\[ b \in \{0, \ldots, 2^n - 1\} \]

**cloud file**

| 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
**Randomized Algorithm**

**local file**

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\[ a \in \{0, \ldots, 2^n - 1\} \]

- Pick random prime \( p \in \{2, 3, \ldots, 100n^2 \log n\} \)

- \( a \mod p \)

- EQ iff \( a = b \mod p \)

- \( b \in \{0, \ldots, 2^n - 1\} \)

**cloud file**

| 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
ANALYSIS

• If \( a = b \), then for every \( p \), \( a = b \mod p \). We always output **EQ**!

• If \( a \neq b \), how often do we output **EQ**?

• \( a - b = 0 \mod p \).

\[ 2^n \geq a - b = \prod_{i=1}^{k} p_i \]

• Prime Number Theorem: there are \( \approx \frac{N}{\log N} \) prime numbers in the interval \( \{2, 3, \ldots, N\} \)

• With probability \( \approx 1 - \frac{1}{100} \), the output is correct
ANALYSIS

• If $a = b$, then for every $p$, $a = b \mod p$. We always output $EQ$!
ANALYSIS

- If $a = b$, then for every $p$, $a = b \mod p$. We always output $EQ$!
- If $a \neq b$, how often do we output $EQ$?
ANALYSIS

• If \( a = b \), then for every \( p \), \( a = b \mod p \). We always output \( EQ! \).

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ANALYSIS

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  $2^n \geq a - b$
ANALYSIS

• If \( a = b \), then for every \( p \), \( a = b \mod p \). We always output EQ!

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2^n \geq a - b = p_1 \cdot p_2 \cdots p_k
\]
• If \( a = b \), then for every \( p \), \( a = b \mod p \). We always output \( EQ! \)
• If \( a \neq b \), how often do we output \( EQ \)?
• \( a - b = 0 \mod p \).
  \[ 2^n \geq a - b = p_1 \cdot p_2 \cdots p_k \geq 2^k \]
Analysis

• If $a = b$, then for every $p$, $a = b \mod p$. We always output $EQ$!

• If $a \neq b$, how often do we output $EQ$?

• $a - b = 0 \mod p$.

  \[2^n \geq a - b = p_1 \cdot p_2 \cdots p_k \geq 2^k\]

• Prime Number Theorem: there are $\approx N / \log N$ prime numbers in the interval $\{2, 3, \ldots, N\}$
ANALYSIS

• If \( a = b \), then for every \( p \), \( a = b \mod p \). We always output EQ!

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LINEARITY OF EXPECTATION

\[ E[X + Y]? \]
LINEARITY OF EXPECTATION

\[ \mathbb{E}[X + Y] = \sum_{i,j} \Pr[X = x_i \cap Y = y_j] \cdot (x_i + y_j) \]
LINEARITY OF EXPECTATION

\[ \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] \]

\[ \mathbb{E}[X + Y] = \sum_{i,j} \Pr[X = x_i \cap Y = y_j] \cdot (x_i + y_j) \]

\[ = \sum_{i} x_i \sum_{j} \Pr[X = x_i \cap Y = y_j] \]

\[ + \sum_{j} y_j \sum_{i} \Pr[X = x_i \cap Y = y_j] \]
LINEARITY OF EXPECTATION

\( \mathbb{E}[X + Y]? \)

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\mathbb{E}[X + Y] = \sum_{i,j} \text{Pr}[X = x_i \cap Y = y_j] \cdot (x_i + y_j)
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= \sum_i x_i \text{Pr}[X = x_i] + \sum_j y_j \text{Pr}[Y = y_j]
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**LINEARITY OF EXPECTATION**

$$\mathbb{E}[X + Y] = \sum_{i,j} \Pr[X = x_i \cap Y = y_j] \cdot (x_i + y_j)$$

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$$= \sum_i x_i \Pr[X = x_i] + \sum_j y_j \Pr[Y = y_j]$$

$$= \mathbb{E}[X] + \mathbb{E}[Y]$$
LINEARITY OF EXPECTATION

• One die: \( \mathbb{E}[X] = 3.5 \)
LINEARITY OF EXPECTATION

• One die: $\mathbb{E}[X] = 3.5$

• Five dice? $\mathbb{E}[X_1 + X_2 + X_3 + X_4 + X_5]$?
LINEARITY OF EXPECTATION

• One die: \( \mathbb{E}[X] = 3.5 \)

• Five dice? \( \mathbb{E}[X_1 + X_2 + X_3 + X_4 + X_5] \) ?

• By linearity of expectation:

\[
\mathbb{E}[X_1 + X_2 + X_3 + X_4 + X_5] \\
= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + \mathbb{E}[X_4] + \mathbb{E}[X_5] \\
= 5 \cdot 3.5 = 17.5
\]
• Alice and Bob have (unusual) dice
• Numbers on Alice’s die are 2, 2, 2, 2, 3, 3
• Numbers on Bob’s die are 1, 1, 1, 1, 6, 6
• Alice and Bob throw their dice; the one with the larger number on the die wins
• Whose die has larger expected number?
• Who wins with higher probability?
Maximum Cut (Max-CUT)
Maximum Cut

- Undirected graph $G$, vertices $V$, edges $E$
Maximum Cut

- Undirected graph $G$, vertices $V$, edges $E$
- Bipartition of $V$ that maximizes the number of edges crossing the partition

NP-hard to solve
Maximum Cut

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- Bipartition: $S \subseteq V$, $\overline{S} \subseteq V$
**Maximum Cut**

- Undirected graph $G$, vertices $V$, edges $E$
- Bipartition of $V$ that maximizes the number of edges crossing the partition
- Bipartition: $S \subseteq V, \overline{S} \subseteq V$
- Cut $\delta(S) = \{(u, v) \in E: u \in S, v \in \overline{S}\}$
- \text{Max-CUT}: $\max_{S \subseteq V} \delta(S)$
- NP-hard to solve
Maximum Cut

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- **NP**-hard to solve
Maximum Cut

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- NP-hard to solve exactly
RANDOMIZED APPROXIMATION

• Output a random subset $S \subseteq V$
**Randomized Approximation**

- Output a random subset $S \subseteq V$
- In other words, add each vertex $v$ in $S$ independently with probability $1/2$
Randomized Approximation

- Output a random subset $S \subseteq V$

- In other words, add each vertex $v$ in $S$ independently with probability $1/2$

- Each edge $(u, v)$ is cut with probability $1/2$
ANALYSIS

• $X_{u,v} = 1$ if $(u, v)$ is cut, $X_{u,v} = 0$ otherwise
ANALYSIS

• $X_{u,v} = 1$ if $(u, v)$ is cut, $X_{u,v} = 0$ otherwise
• $X_{u,v} = 1$ with probability $1/2$

Number of cut edges

$\sum_{(u,v) \in E} X_{u,v}$

Expected number of cut edges

$E \left[ \sum_{(u,v) \in E} X_{u,v} \right] = \sum_{(u,v) \in E} E \left[ X_{u,v} \right] = |E| / 2$
• $X_{u,v} = 1$ if $(u, v)$ is cut, $X_{u,v} = 0$ otherwise
• $X_{u,v} = 1$ with probability $1/2$
• $\mathbb{E}[X_{u,v}] = 1/2$
ANALYSIS

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• $\mathbb{E}[X_{u,v}] = 1/2$
• Number of cut edges

$$\sum_{(u,v) \in E} X_{u,v}$$

• Expected number of cut edges

$$\mathbb{E} \left[ \sum_{(u,v) \in E} X_{u,v} \right] = \sum_{(u,v) \in E} \mathbb{E}[X_{u,v}] = |E|/2$$
2-APPROXIMATION

- Max-CUT: $\text{OPT} \leq |E|$
2-APPROXIMATION

- Max-CUT: $\text{OPT} \leq |E|$
- Our algorithm: $\mathbb{E}[\delta(S)] \geq |E|/2$
2-APPROXIMATION

• Max-CUT: $\text{OPT} \leq |E|$

• Our algorithm: $\mathbb{E}[\delta(S)] \geq |E|/2$

• $\mathbb{E}[\delta(S)] \geq \text{OPT} /2$
2-APPROXIMATION

- Max-CUT: $\text{OPT} \leq |E|$

- Our algorithm: $\mathbb{E}[\delta(S)] \geq |E|/2$

- $\mathbb{E}[\delta(S)] \geq \text{OPT} / 2$

- Can we have algorithm that always outputs $\delta(S) \geq \text{OPT} / 2$?
Markov’s Inequality

Theorem

If $X$ is a non-negative random variable*, then

$$\forall a, \quad \Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a}.$$
**Theorem**

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$$\forall a, \quad \Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a}.$$ 

**Examples:**

$$\Pr[X \geq 2\mathbb{E}[X]] \leq \frac{1}{2}.$$
Markov’s Inequality

Theorem

If $X$ is a non-negative random variable*, then

$$\forall a, \quad \Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a}.$$  

Examples:

$$\Pr[X \geq 2\mathbb{E}[X]] \leq \frac{1}{2}.$$  

$$\Pr[X \geq 5\mathbb{E}[X]] \leq \frac{1}{5}.$$
<table>
<thead>
<tr>
<th><strong>Problem</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>A lottery ticket costs 10 dollars. A 40% of a lottery budget goes to prizes. Show that the chances to win 500 dollars or more are less than 1%</td>
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<td>LOTTERY BUDGET</td>
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- Assume the contrary: the probability to win 500 dollars or more is at least 0.01
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- Denote the number of tickets sold by \( n \)
- Then the budget of the lottery is 10\( n \) dollars
A lottery ticket costs 10 dollars. A 40% of a lottery budget goes to prizes. Show that the chances to win 500 dollars or more are less than 1%

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- Then the budget of the lottery is $10n$ dollars
- $10n \times 0.4 = 4n$ dollars are spent on the prizes
- By our assumption at least $\frac{n}{100}$ tickets win at least 500 dollars
**LOTTERY BUDGET**

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- Contradiction!
GEOMETRIC PROOF

\[ E[X] \geq a \times \Pr[X \geq a] \]
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The gray region is larger; the inequality follows.
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Approximation Guarantee

- $\mathbb{E}[\# \text{cut edges}] = |E|/2 \implies \mathbb{E}[\# \text{uncut edges}] = |E|/2$

With probability at least $\varepsilon/2$, we have a $2 - \varepsilon$-approximation.

Ex. $\varepsilon = 1/100$: with probability at least $1/200$, we have a $2.03$-approximation.
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Example: $\varepsilon = 1/100$: with probability at least $1/200$, we have a $2/1.03$-approximation.
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Ex. \( \varepsilon = \frac{1}{100} \): with probability at least \( \frac{1}{200} \), we have 0.03-approximation
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PROBABILITY AMPLIFICATION

• Pick independent uniform subsets $S_1, \ldots, S_k \subseteq V$
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We have $2^{1 - \varepsilon}$-approximation with probability $\frac{1}{10}$. 

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  \leq (1 - \varepsilon/2)^k \leq e^{-\varepsilon k/2} \leq \frac{1}{10^{10}n} \text{ for } k = \frac{2 \ln n + 50}{\varepsilon}
  \]
- We have $\frac{2}{1-\varepsilon}$-approximation with probability $1 - \frac{1}{10^{10}n}$
Summary

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• We can amplify probability of success by independent repetitions