PREVIOUSLY...

• Exact Algorithms

• Randomized Algorithms

• Approximate Algorithms
Previously...

- Exact Algorithms
- Randomized Algorithms
- Approximate Algorithms
- Today: More examples
Map Coloring
THE LAND OF OZ
SWISS CANTONS
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- Computer checked almost 2000 graphs.

Theorem [Weak Version]

Every map can be colored with 6 colors.
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## Six Color Theorem

### Theorem [Weak Version]

Every map can be colored with 6 colors.

- **Induction** on the number of countries $n$. 

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- **Induction assumption.** All maps with $k$ countries can be colored with 6 colors.
Theorem [Weak Version]

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- **Induction** on the number of countries $n$.
- **Base case.** $n \leq 6$: can color with 6 colors.
- **Induction assumption.** All maps with $k$ countries can be colored with 6 colors.
- **Induction step.** We’ll show that any map with $k + 1$ countries can be colored with 6 colors.
### Lemma

Every map contains a country $v$ with at most 5 neighbors.
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Graph Coloring
A graph coloring is a coloring of the graph vertices s.t. no pair of adjacent vertices share the same color.
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• The chromatic number $\chi(G)$ of a graph $G$ is the smallest number of colors needed to color the graph.
CHROMATIC NUMBER

Chromatic number is 3
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Chromatic Number

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Chromatic number is 3
The chromatic number of $K_n$ is $n$. 
For $n > 1$, the chromatic number of $P_n$ is 2.
For even $n$, the chromatic number of $C_n$ is 2.
For odd $n > 2$, the chromatic number of $C_n$ is 3.
The chromatic number of a bipartite graph (with at least 1 edge) is 2.
Applications
EXAM SCHEDULE

• Each student takes an exam in each of her courses
• All students in one course take the exam together
• One student cannot take two exams per day
• What is the minimum number of days needed for the exams?
Exam Schedule

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Graphs

Proofs

Numbers

Combs

Project
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Bandwidth allocation

Different stations are allowed to use the same frequency if they are far apart. What is an optimal assignment of frequencies to stations?
OTHER APPLICATIONS

• Scheduling Problems
• Register Allocation
• Sudoku puzzles
• Taxis scheduling
• ...

...
Exact Algorithm for Coloring
Dynamic Programming

• Given graph $G$ on $n$ vertices, find $\chi(G)$—minimum number of colors in a valid coloring of $G$
Dynamic Programming

- Given graph $G$ on $n$ vertices, find $\chi(G)$—minimum number of colors in a valid coloring of $G$
- Dynamic programming is one of the most powerful algorithmic techniques
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• Dynamic programming is one of the most powerful algorithmic techniques

• Rough idea: express a solution for a problem through solutions for smaller subproblems
Subproblems

• For a subset of vertices $S \subseteq \{1, \ldots, n\}$ compute $\chi(S)$—the minimum number of colors needed to color vertices $S$. 
SUBPROBLEMS

• For a subset of vertices $S \subseteq \{1, \ldots, n\}$ compute $\chi(S)$ — the minimum number of colors needed to color vertices $S$.

• Consider $S$. For any subset $U \subseteq S$, if there are no edges between vertices from $U$, we can color them all in one color, and use $\chi(S \setminus U)$ to color the rest. $\chi(S) = \min_{U \text{ without edges}} 1 + \chi(S \setminus U)$.
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$$\chi(S) = \min_{U \text{ without edges}} \left( 1 + \chi(S \setminus U) \right)$$
ORDER OF SUBPROBLEMS

• Need to process all subsets $S \subseteq \{1, \ldots, n\}$ in order that guarantees that when computing the value of $\chi(S)$, the values of $\chi(S \setminus U)$ have already been computed
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• Need to process all subsets $S \subseteq \{1, \ldots, n\}$ in order that guarantees that when computing the value of $\chi(S)$, the values of $\chi(S \setminus U)$ have already been computed.
• For example, we can process subsets in order of increasing size.
\( \chi(\emptyset) = 0 \)
Algorithm

\[ \chi(\emptyset) = 0 \]

for \( s \) from 1 to \( n \):

\[ \text{for all } S \subseteq \{1, \ldots, n\} \text{ of size } s: \]
$\chi(\emptyset) = 0$

for s from 1 to n:

for all $S \subseteq \{1, \ldots, n\}$ of size s:

for all $U \subseteq S$, $U$ without edges

$\chi(S) \leftarrow \min\{\chi(S), \chi(S \setminus U) + 1\}$

return $\chi(\{1, \ldots, n\})$
Algorithm

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for \( s \) from 1 to \( n \):

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Randomized Algorithm for 3-Coloring
RANDOMIZED ALGORITHM

• Given a 3-colorable graph, find a 3-coloring
Randomized Algorithm

• Given a 3-colorable graph, find a 3-coloring

• This problem is **NP**-hard, we’ll give an exponential-time algorithm
RANDOMIZED ALGORITHM

- Forbid one random color at each vertex

Solve 2-SAT in polynomial time. Repeat the algorithm \((3/2)^n\) times.
RANDOMIZED ALGORITHM

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RANDOMIZED ALGORITHM

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Approximate Algorithm for 3-Coloring
Approximate Coloring

• Given a 3-colorable graph, finding a 3-coloring is \textbf{NP-hard}
Approximate Coloring

• Given a 3-colorable graph, finding a 3-coloring is \textit{NP}-hard

• Given a 3-colorable graph, finding an \textit{n}-coloring is \textit{trivial}
Approximate Coloring

• Given a 3-colorable graph, finding a 3-coloring is \textbf{NP-hard}

• Given a 3-colorable graph, finding an \textit{n}-coloring is \textit{trivial}

• We’ll see how to find an \textit{O} (\sqrt{n})-coloring in \textit{polynomial} time
# Graphs of Bounded Degree

## Greedy Coloring

A graph $G$ where each vertex has degree $\leq \Delta$ can be colored with $\Delta + 1$ colors.
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A graph $G$ where each vertex has degree $\leq \Delta$ can be colored with $\Delta + 1$ colors.
APPROXIMATE ALGORITHM

While there is vertex $v \in G$ of degree $\geq \sqrt{n}$:

Color the neighbors of $v$ in 2 new colors, remove them from the graph.

All remaining vertices have degree $< \sqrt{n}$. Color the rest of the graph using $\sqrt{n}$ new colors.
While there is vertex $v \in G$ of degree $\geq \sqrt{n}$:

- Color the neighbors of $v$ in 2 new colors, remove them from the graph.
- All remaining vertices have degree $< \sqrt{n}$.
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While there is vertex \( v \in G \) of degree \( \geq \sqrt{n} \):

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- All remaining vertices have degree \( < \sqrt{n} \).
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![Diagram of a graph with a vertex and its neighbors colored in different colors.](image-url)
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ANALYSIS