GEMS OF TCS

HEURISTIC ALGORITHMS

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HEURISTIC ALGORITHMS

• When exact algorithms are too slow, and approximate algorithm are not accurate enough

• Heuristic algorithms use practical methods that are not guaranteed/proved to be optimal or efficient

• Some heuristic algorithms are fast but not guaranteed to find optimal solutions

• Some heuristic algorithms find optimal solutions but not guaranteed to be fast
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Heuristic Algorithms

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- We can use **heuristic** algorithms
- **Heuristic** algorithms use practical methods that are not guaranteed/proved to be optimal or efficient
- Some heuristic algorithms are fast but not guaranteed to find optimal solutions
- Some heuristic algorithms find optimal solutions but not guaranteed to be fast
Traveling Salesman
TRAVELING SALESMAN PROBLEM

Given a complete weighted graph, find a cycle (or a path) of minimum total weight (length) visiting each node exactly once.
TRAVELING SALESMAN PROBLEM

Given a complete weighted graph, find a cycle (or a path) of minimum total weight (length) visiting each node exactly once

length: 9
Nearest Neighbors

- Going to the nearest unvisited node at every iteration?
NEAREST NEIGHBORS

• Going to the nearest unvisited node at every iteration?
• Efficient, works reasonably well in practice
Nearest Neighbors

• Going to the nearest unvisited node at every iteration?
• Efficient, works reasonably well in practice
• May produce a cycle that is much worse than an optimal one
Nearest Neighbors: Bad Case

- How to fool the nearest neighbors heuristic?
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- Assume that the weights of almost all the edges in the graph are equal to 2
NEAREST NEIGHBORS: BAD CASE

• How to fool the nearest neighbors heuristic?
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• And we start to construct a cycle:
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• And we start to construct a cycle:
Suboptimal Solution for Euclidean TSP
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$\text{OPT} \approx 26.42$
Suboptimal Solution for Euclidean TSP

OPT ≈ 26.42
SUBOPTIMAL SOLUTION FOR EUCLIDEAN TSP

OPT \approx 26.42
Suboptimal Solution for Euclidean TSP

$OPT \approx 26.42$
Suboptimal Solution for Euclidean TSP

$$\text{OPT} \approx 26.42$$
Suboptimal Solution for Euclidean TSP

OPT ≈ 26.42
NN ≈ 28.33
For Euclidean instances, the resulting cycle is $O(\log n)$-approximate.
LOCAL SEARCH

• $s \leftarrow$ some initial solution
Local Search

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- while it is possible to change 2 edges in $s$ to get a better cycle $s'$:
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- while it is possible to change 2 edges in $s$ to get a better cycle $s'$:
  - $s \leftarrow s'$
LOCAL SEARCH

• \( s \leftarrow \text{some initial solution} \)
• while it is possible to change 2 edges in \( s \) to get a better cycle \( s' \):
  • \( s \leftarrow s' \)
• return \( s \)
Changing two edges in a suboptimal solution:
Example

Changing two edges in a suboptimal solution:
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Changing two edges in a suboptimal solution:
A suboptimal solution that cannot be improved by changing two edges:
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Need to allow changing three edges to improve this solution
Local Search with parameter $d$:

- $s \leftarrow$ some initial solution
- while it is possible to change $d$ edges in $s$ to get a better cycle $s'$:
  - $s \leftarrow s'$
- return $s$
Properties

- Computes a local optimum instead of a global optimum
Properties

- Computes a local optimum instead of a global optimum
- The larger $d$, the better the resulting solution and the higher is the running time
Performance

- Trade-off between quality and running time of a single iteration
Performance

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- Still, the number of iterations may be exponential and the quality of the found cycle may be poor
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- Trade-off between quality and running time of a single iteration
- Still, the number of iterations may be exponential and the quality of the found cycle may be poor
- But works well in practice
Satisfiability
\[(x_1 \lor x_2 \lor x_3) \land (x_1 \lor \lnot x_2) \land (\lnot x_1 \lor x_3) \land (x_2 \lor \lnot x_3)\]
\[ (x_1 \lor x_2 \lor x_3) \land (x_1 \lor \neg x_2) \land (\neg x_1 \lor x_3) \land (x_2 \lor \neg x_3) \land (\neg x_1 \lor \neg x_2 \lor \neg x_3) \]
BACKTRACKING

- Construct a solution piece by piece
BACKTRACKING

• Construct a solution piece by piece
• Backtrack if the current partial solution cannot be extended to a valid solution
\[(x_1 \lor x_2 \lor x_3 \lor x_4)(\neg x_1)(x_1 \lor x_2 \lor \neg x_3)(x_1 \lor \neg x_2)(x_2 \lor \neg x_4)\]
\((x_1 \lor x_2 \lor x_3 \lor x_4)(\neg x_1)(x_1 \lor x_2 \lor \neg x_3)(x_1 \lor \neg x_2)(x_2 \lor \neg x_4)\)

\(x_1 = 0\)

\((x_2 \lor x_3 \lor x_4)(x_2 \lor \neg x_3)(\neg x_2)(x_2 \lor \neg x_4)\)
Example

\[(x_1 \lor x_2 \lor x_3 \lor x_4)(\neg x_1)(x_1 \lor x_2 \lor \neg x_3)(x_1 \lor \neg x_2)(x_2 \lor \neg x_4)\]

\(x_1 = 0\)

\[(x_2 \lor x_3 \lor x_4)(x_2 \lor \neg x_3)(\neg x_2)(x_2 \lor \neg x_4)\]

\(x_2 = 0\)

\[(x_3 \lor x_4)(\neg x_3)(\neg x_4)\]
Example

\[
(x_1 \lor x_2 \lor x_3 \lor x_4)(\neg x_1)(x_1 \lor x_2 \lor \neg x_3)(x_1 \lor \neg x_2)(x_2 \lor \neg x_4)
\]

\[
(x_2 \lor x_3 \lor x_4)(x_2 \lor \neg x_3)(\neg x_2)(x_2 \lor \neg x_4)
\]

\[
(x_3 \lor x_4)(\neg x_3)(\neg x_4)
\]

\[
(x_4)(\neg x_4)
\]
(x_1 \lor x_2 \lor x_3 \lor x_4)(\neg x_1)(x_1 \lor x_2 \lor \neg x_3)(x_1 \lor \neg x_2)(x_2 \lor \neg x_4)

x_1 = 0

(x_2 \lor x_3 \lor x_4)(x_2 \lor \neg x_3)(\neg x_2)(x_2 \lor \neg x_4)

x_2 = 0

(x_3 \lor x_4)(\neg x_3)(\neg x_4)

x_3 = 0

(x_4)(\neg x_4)

x_4 = 0

()
(x_1 \lor x_2 \lor x_3 \lor x_4)(\neg x_1)(x_1 \lor x_2 \lor \neg x_3)(x_1 \lor \neg x_2)(x_2 \lor \neg x_4)

x_1 = 0

(x_2 \lor x_3 \lor x_4)(x_2 \lor \neg x_3)(\neg x_2)(x_2 \lor \neg x_4)

x_2 = 0

(x_3 \lor x_4)(\neg x_3)(\neg x_4)

x_3 = 0

(x_4)(\neg x_4)

x_3 = 1

()()\neg x_4)

x_4 = 0

()()\neg x_4)

x_4 = 1

()()
$$\left( x_1 \lor x_2 \lor x_3 \lor x_4 \right) \left( \neg x_1 \right) \left( x_1 \lor x_2 \lor \neg x_3 \right) \left( x_1 \lor \neg x_2 \right) \left( x_2 \lor \neg x_4 \right)$$
\[(x_1 \lor x_2 \lor x_3 \lor x_4)(\neg x_1)(x_1 \lor x_2 \lor \neg x_3)(x_1 \lor \neg x_2)(x_2 \lor \neg x_4)\]

\(x_1 = 0\)

\[(x_2 \lor x_3 \lor x_4)(x_2 \lor \neg x_3)(\neg x_2)(x_2 \lor \neg x_4)\]

\(x_1 = 1\)

\(x_2 = 0\)

\[(x_3 \lor x_4)(\neg x_3)(\neg x_4)\]

\(x_2 = 1\)

\(x_3 = 0\)

\[(x_4)(\neg x_4)\]

\(x_3 = 1\)

\[(\neg x_4)\]

\(x_4 = 0\)

\(x_4 = 1\)
BACKTRACKING ALGORITHM

- SolveSAT($F$):
  - if $F$ has no clauses:
    return "sat"
  - if $F$ contains an empty clause:
    return "unsat"
BACKTRACKING ALGORITHM

• SolveSAT($F$):
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  • $x \leftarrow$ unassigned variable of $F$
Backtracking Algorithm

- **SolveSAT**(*F*):
  - if *F* has no clauses:
    return “sat”
  - if *F* contains an empty clause:
    return “unsat”
  - *x* ← unassigned variable of *F*
  - if **SolveSAT**( *F*[ *x* ← 0 ]) = “sat”:
    return “sat”
Backtracking Algorithm

- **SolveSAT**(\(F\)):
  - if \(F\) has no clauses:
    return “sat”
  - if \(F\) contains an empty clause:
    return “unsat”
  - \(x \leftarrow\) unassigned variable of \(F\)
  - if **SolveSAT**(\(F[x \leftarrow 0]\)) = “sat”:
    return “sat”
  - if **SolveSAT**(\(F[x \leftarrow 1]\)) = “sat”:
    return “sat”
BACKTRACKING ALGORITHM

- **SolveSAT**(F):
  - if F has no clauses:
    return “sat”
  - if F contains an empty clause:
    return “unsat”
  - x ← unassigned variable of F
  - if SolveSAT(F[x ← 0]) = “sat”:
    return “sat”
  - if SolveSAT(F[x ← 1]) = “sat”:
    return “sat”
  - return “unsat”
• Thus, instead of considering all $2^n$ branches of the recursion tree, we track carefully each branch
• Thus, instead of considering all $2^n$ branches of the recursion tree, we track carefully each branch

• When we realize that a branch is dead (cannot be extended to a solution), we immediately cut it
SAT Solvers

- Backtracking is used in many state-of-the-art SAT-solvers
SAT SOLVERS

• Backtracking is used in many state-of-the-art SAT-solvers
• SAT-solvers use tricky heuristics to choose a variable to branch on, simplify a formula before branching, and use efficient data structures
SAT Solvers

- Backtracking is used in many state-of-the-art SAT-solvers.
- SAT-solvers use tricky heuristics to choose a variable to branch on, simplify a formula before branching, and use efficient data structures.
- Another commonly used technique is local search.
Applications
THE ART OF COMPUTER PROGRAMMING

VOLUME 4  PRE-FASCICLE 6A

A DRAFT OF
SECTION 7.2.2.2:
SATISFIABILITY

DONALD E. KNUTH  Stanford University
Wow! — Section 7.2.2.2 has turned out to be the longest section, by far, in *The Art of Computer Programming*. The SAT problem is evidently a “killer app,” because it is key to the solution of so many problems. Consequently I can only hope that my lengthy treatment does not also kill off my faithful readers!

Donald Knuth
• Annual SAT Conference (since 1996):
  http://satisfiability.org
• Annual SAT Conference (since 1996): http://satisfiability.org
• Annual SAT Solving competitions (since 2002): http://www.satcompetition.org/
CONFERENCE, COMPETITION, JOURNAL

• Annual SAT Conference (since 1996): http://satisfiability.org
• Annual SAT Solving competitions (since 2002): http://www.satcompetition.org/
• Journal on Satisfiability, Boolean Modeling and Computation: http://jsatjournal.org/
Two-hundred-terabyte maths proof is largest ever

A computer cracks the Boolean Pythagorean triples problem — but is it really maths?

Evelyn Lamb

26 May 2016
Computer Search Settles 90-Year-Old Math Problem

By translating Keller’s conjecture into a computer-friendly search for a type of graph, researchers have finally resolved a problem about covering spaces with tiles.
from pycosat import solve

clauses = [ [-1, -2, -3], [1, -2], [2, -3], [3, -1], [1, 2, 3] ]

print(solve(clauses))
print(solve(clauses[1:]))
from pycosat import solve

clauses = [ [-1, -2, -3], [1, -2], [2, -3], [3, -1], [1, 2, 3] ]

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UNSAT
[1, 2, 3]
Is it possible to place $n$ queens on an $n \times n$ board such that no two of them attack each other?
Examples
ENCODING AS SAT

• $n^2$ 0/1-variables: for $0 \leq i, j < n$, $x_{ij} = 1$ iff queen is placed into cell $(i,j)$
ENCODING AS SAT

• $n^2$ 0/1-variables: for $0 \leq i, j < n$, $x_{ij} = 1$ iff queen is placed into cell $(i, j)$
• For $0 \leq i < n$, $i$th row contains $\geq 1$ queen:
  $$(x_{i0} = 1 \text{ or } x_{i2} = 1 \text{ or } \ldots \text{ or } x_{i(n-1)} = 1).$$
ENCODING AS SAT

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- For $0 \leq i < n$, $i$th row contains $\geq 1$ queen:
  $$(x_{i0} = 1 \text{ or } x_{i2} = 1 \text{ or } \ldots \text{ or } x_{i(n-1)} = 1).$$
- For $0 \leq i < n$, $i$th row contains $\leq 1$ queen:
  $$\forall 0 \leq j_1 \neq j_2 < n: \ (x_{ij_1} = 0 \text{ or } x_{ij_2} = 0).$$
ENCODING AS SAT

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- For $0 \leq i < n$, $i$th row contains $\geq 1$ queen:
  $$(x_{i0} = 1 \text{ or } x_{i2} = 1 \text{ or } \ldots \text{ or } x_{i(n-1)} = 1).$$
- For $0 \leq i < n$, $i$th row contains $\leq 1$ queen:
  $$\forall 0 \leq j_1 \neq j_2 < n: (x_{ij_1} = 0 \text{ or } x_{ij_2} = 0).$$
- For $0 \leq j < n$, $j$th column contains $\leq 1$ queen:
  $$\forall 0 \leq i_1 \neq i_2 < n: (x_{i_1j} = 0 \text{ or } x_{i_2j} = 0).$$
ENCODING AS SAT

- \( n^2 \) 0/1-variables: for \( 0 \leq i, j < n \), \( x_{ij} = 1 \) iff a queen is placed into cell \((i, j)\).
- For \( 0 \leq i < n \), \( i \)th row contains \( \geq 1 \) queen:
  \[ (x_{i0} = 1 \text{ or } x_{i2} = 1 \text{ or } \ldots \text{ or } x_{i(n-1)} = 1). \]
- For \( 0 \leq i < n \), \( i \)th row contains \( \leq 1 \) queen:
  \[ \forall 0 \leq j_1 \neq j_2 < n: \ (x_{ij_1} = 0 \text{ or } x_{ij_2} = 0). \]
- For \( 0 \leq j < n \), \( j \)th column contains \( \leq 1 \) queen:
  \[ \forall 0 \leq i_1 \neq i_2 < n: \ (x_{i_1j} = 0 \text{ or } x_{i_2j} = 0). \]
- For each pair \((i_1, j_1), (i_2, j_2)\) on diagonal:
  \[ (x_{i_1j_1} = 0 \text{ or } x_{i_2j_2} = 0). \]
```python
from itertools import combinations, product
from pycosat import solve

n = 10
clauses = []

# converts a pair of integers into a unique integer
def varnum(i, j):
    assert i in range(n) and j in range(n)
    return i * n + j + 1

# each row contains at least one queen
for i in range(n):
    clauses.append([varnum(i, j) for j in range(n)])

# each row contains at most one queen
for i in range(n):
    for j1, j2 in combinations(range(n), 2):
        clauses.append([-varnum(i, j1), -varnum(i, j2)])

# each column contains at most one queen
for j in range(n):
    for i1, i2 in combinations(range(n), 2):
        clauses.append([-varnum(i1, j), -varnum(i2, j)])

# no two queens stay on the same diagonal
for i1, j1, i2, j2 in product(range(n), repeat=4):
    if i1 == i2:
        continue
    if abs(i1 - i2) == abs(j1 - j2):
        clauses.append([-varnum(i1, j1),
                         -varnum(i2, j2)])

assignment = solve(clauses)
for i, j in product(range(n), repeat=2):
    if assignment[varnum(i, j) - 1] > 0:
        print(j, end=' ')
```