Gems of TCS

Easy and Hard Problems

Sasha Golovnev

August 24, 2021

Theoretical Computer Science

$P \implies Q$

Mathematical logic

Theoretical Computer Science

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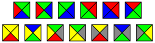
Mathematical logic



Computability theory

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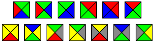

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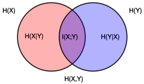

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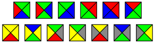

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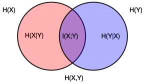

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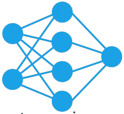

Learning, neural nets

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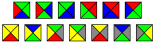

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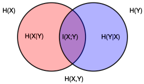

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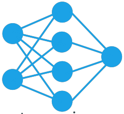

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$$P = NP?$$

Computational complexity

THEORETICAL COMPUTER SCIENCE

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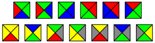

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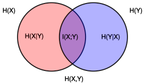

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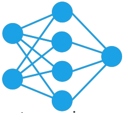

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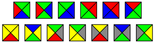

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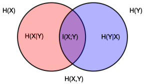

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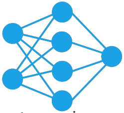

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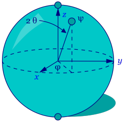

Quantum Algorithms

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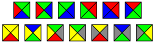

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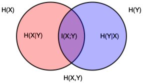

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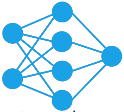

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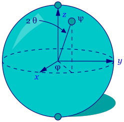

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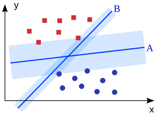

Machine learning

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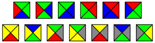

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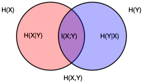

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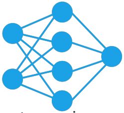

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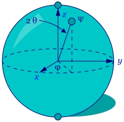

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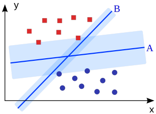

Machine learning

Data Science

This Course

- Theoretical/Mathematical viewpoint

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- Theoretical/Mathematical viewpoint
- Topic overview

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    - Algorithms

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Administrative Info

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- email: `alexgolovnev+gems@gmail.com`

Course Begins

- Running time of an algorithm

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  - $100n^2$ vs $n^3/10$

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- Complexity class P: Problems whose solution can be found efficiently

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- Complexity class NP: Problems whose solution can be verified efficiently

The main open problem in Computer Science

Is P equal to NP?

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Is P equal to NP?

Millenium Prize Problem

Clay Mathematics Institute: $1M prize for solving the problem

- If P=NP, then all NP-problems can be solved in polynomial time.

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- If P≠NP, then there exist NP-problems that cannot be solved in polynomial time.

NP-complete Problems

- The “hardest” problems in NP

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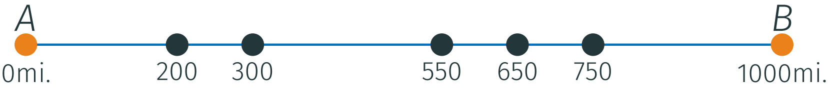
- If one NP-complete problem cannot be solved in polynomial time, then all NP-complete problems cannot be solved in polynomial time
- Later we’ll show NP-complete problems exist!

Car Fueling

Car Fueling

Distance with full tank 300 mi.

Minimize the number of stops at gas stations



Break http://bit.ly/car-fueling

Example

Distance with full tank 300 mi.

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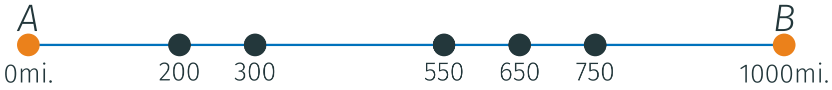

Car Fueling. Solution

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- Runs in linear time $O(n)$, where n is the size of the input (# of gas stations)

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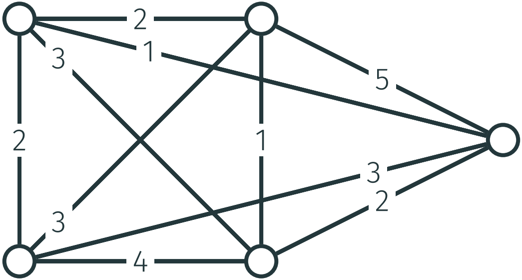
- “Greedy” algorithm
- Runs in linear time $O(n)$, where n is the size of the input (# of gas stations)
- Easy problem

Traveling Salesman Problem (TSP)

Traveling Salesman Problem

Given a complete weighted graph, find a cycle (or a path) of minimum total weight (length) visiting each node exactly once



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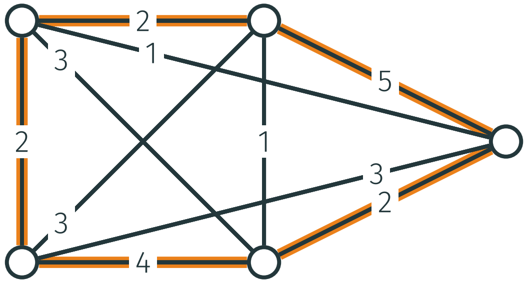

length: 15

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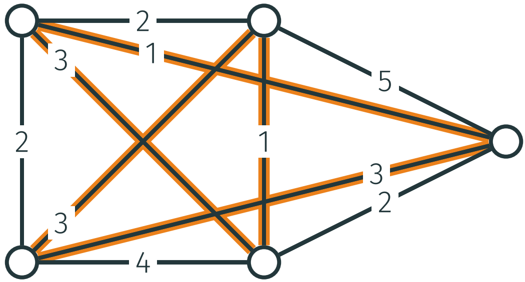

length: 11

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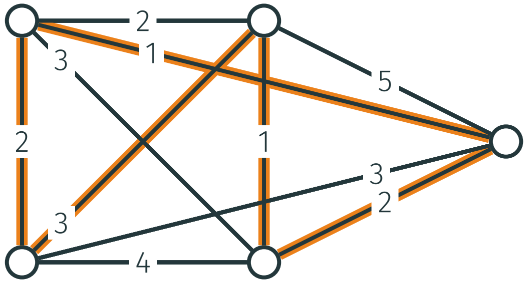

length: 9

Status

- Classical optimization problem with countless number of real life applications (we'll see soon)

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- No polynomial time algorithms known
- The best known algorithm runs in time 2^n

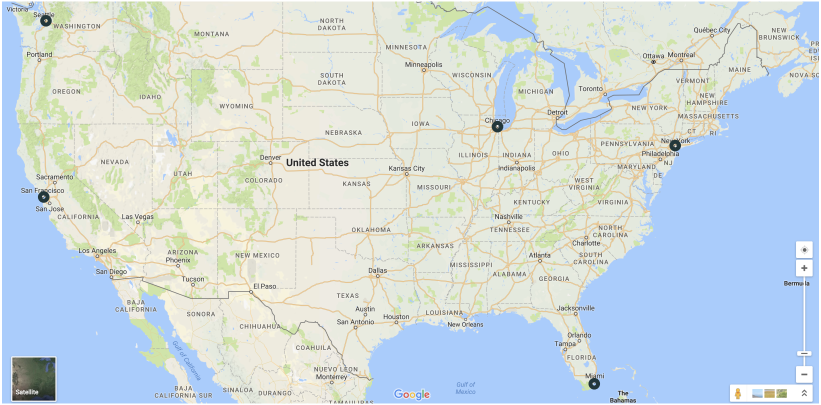
Delivering Goods

Need to visit several points. What is the optimal order of visiting them?

Traveling


United States
Victoria
Seattle
WASHINGTON
MONTANA
NORTH DAKOTA
MINNESOTA
Québec City
NEW BRUNSWICK
Portland
OREGON
IDAHO
WYOMING
SOUTH DAKOTA
Minneapolis
WISCONSIN
MICHIGAN
Toronto
Ottawa
Montreal
MAINE
VERMONT
NEW HAMPSHIRE
NOVA SC
NEW YORK
MASSACHUSETTS
Detroit
Chicago
IOWA
NEBRASKA
CT
RI
New York
PENNSYLVANIA
Philadelphia
NJ
NEVADA
UTAH
Denver
COLORADO
KANSAS
Kansas City
ILLINOIS
INDIANA
Indianapolis
OHIO
MARYLAND
DE
WEST VIRGINIA
VIRGINIA
Sacramento
San Francisco
San Jose
CALIFORNIA
Las Vegas
MISSOURI
KENTUCKY
Nashville
TENNESSEE
NORTH CAROLINA
Charlotte
OKLAHOMA
ARKANSAS
Los Angeles
ARIZONA
Phoenix
NEW MEXICO
SOUTH CAROLINA
Atlanta
MISSISSIPPI
ALABAMA
GEORGIA
San Diego
Tucson
El Paso
Dallas
TEXAS
BAJA CALIFORNIA
SONORA
CHIHUAHUA
Austin
San Antonio
Houston
LOUISIANA
New Orleans
Jacksonville
Orlando
Tampa
FLORIDA
Miami
COAHUILA
NUEVO LEON
Monterrey
Gulf of California
BAJA CALIFORNIA SUR
SINALOA
DURANGO
TAMAULIPAS
Gulf of Mexico
The Bahamas
Satellite
Google

TRAVELING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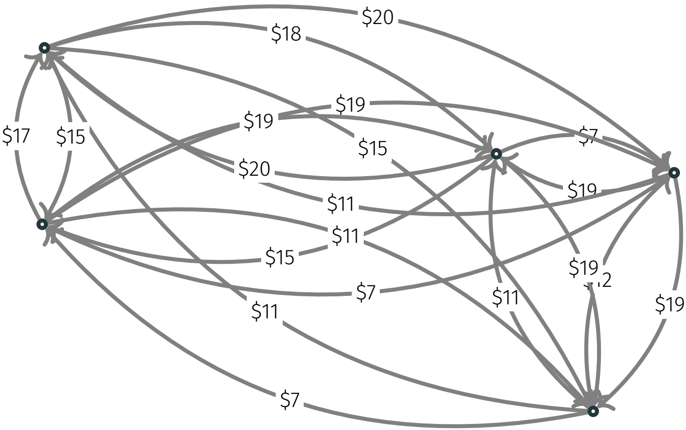
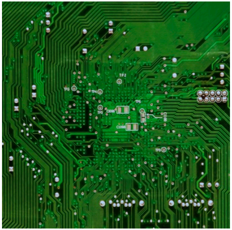


Drilling a Circuit Board



https://developers.google.com/optimization/routing/tsp/tsp

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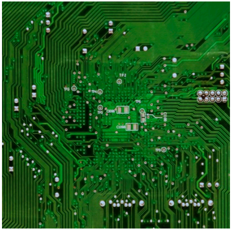

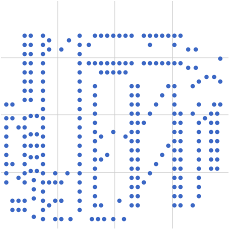

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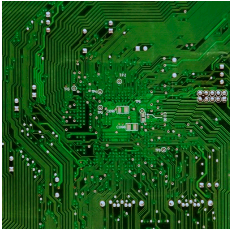

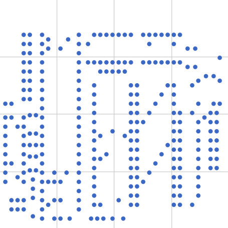

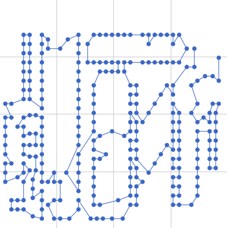

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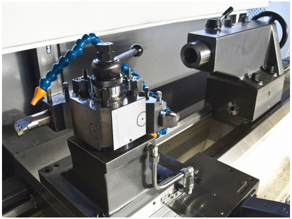
Processing Components

There are n mechanical components to be processed on a complex machine. After processing the i-th component, it takes t_{ij} units of time to reconfigure the machine so that it is able to process the j-th component. What is the minimum processing cost?



Euclidean TSP

- Euclidean TSP: instead of a complete graph, the input consists of n points $p_1 = (x_1, y_1), \ldots, p_n = (x_n, y_n)$ on the plane

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$$d(p_i, p_j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

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- Weights are symmetric: $d(p_i, p_j) = d(p_j, p_i)$

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- Weights are symmetric: $d(p_i, p_j) = d(p_j, p_i)$
- Weights satisfy the triangle inequality: $d(p_i, p_j) \leq d(p_i, p_k) + d(p_k, p_j)$

Brute Force Search

- Finding the best permutation is easy: simply iterate through all of them and select the best one

Brute Force Search

- Finding the best permutation is easy: simply iterate through all of them and select the best one
- But the number of permutations of n objects is $n!$

$n!$: Growth Rate

n	$n!$
5	120
8	40320
10	3628800
13	6227020800
20	2432902008176640000
30	265252859812191058636308480000000

Satisfiability Problem (SAT)

SAT

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3) \wedge (x_2 \vee \neg x_3)$$

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Applications of SAT

- Software Engineering
- Chip testing
- Circuit design
- Automatic theorem provers
- Image analysis
- ...

k-SAT

$$
\begin{aligned}
\phi(x_1, \ldots, x_n) = & (x_1 \vee \neg x_2 \vee \ldots \vee x_k) \quad \wedge \\
& \qquad\qquad \ldots \qquad\qquad\quad \wedge \\
& (x_2 \vee \neg x_3 \vee \ldots \vee x_8)
\end{aligned}
$$

k-SAT

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ϕ is satisfiable if

$$\exists x \in \{0,1\}^n \colon \phi(x) = 1\,.$$

Otherwise, ϕ is unsatisfiable

k-SAT

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\end{aligned}
$$

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k-SAT is SAT where clause length $\leq k$

k-SAT. Examples

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3) \wedge (x_2 \vee \neg x_3)$$

k-SAT. Examples

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3) \wedge (x_2 \vee \neg x_3)$$

$$(x_1) \wedge (\neg x_2) \wedge (x_3) \wedge (\neg x_1)$$

Queen of NP-complete problems

- Cook-Levin Theorem [Coo71, Lev73]: SAT can model non-deterministic Turing machine:

  SAT is NP-complete

Queen of NP-complete problems

- Cook-Levin Theorem [Coo71, Lev73]: SAT can model non-deterministic Turing machine: SAT is NP-complete
- 3-SAT is NP-complete

Queen of NP-complete problems

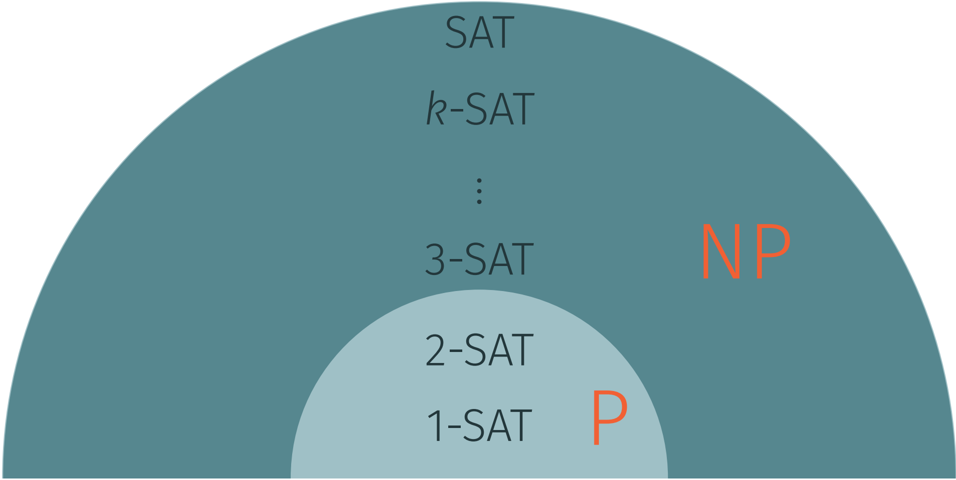
- Cook-Levin Theorem [Coo71, Lev73]: SAT can model non-deterministic Turing machine: SAT is NP-complete

- 3-SAT is NP-complete

- 2-SAT is in P

Complexity of SAT

Complexity of SAT



The SAT game

http://bit.ly/sat-game