Gems of TCS

P vs NP

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Search Problems

Search Problem

Definition

A search problem is defined by an algorithm $\mathcal{C}$ that takes an instance I and a candidate solution S, and runs in time polynomial in the length of I. We say that S is a solution to I iff $\mathcal{C}(S, I) = \texttt{true}$.

SAT

Example

For SAT, I is a Boolean formula, S is an assignment of Boolean constants to its variables. The corresponding algorithm $\mathcal{C}$ checks whether S satisfies all clauses of I.

Class NP

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NP is the class of all search problems.

- **NP** stands for “non-deterministic polynomial time”: one can guess a solution, and then verify its correctness in polynomial time

- NP stands for "non-deterministic polynomial time": one can guess a solution, and then verify its correctness in polynomial time
- In other words, the class NP contains all problems whose solutions can be efficiently verified

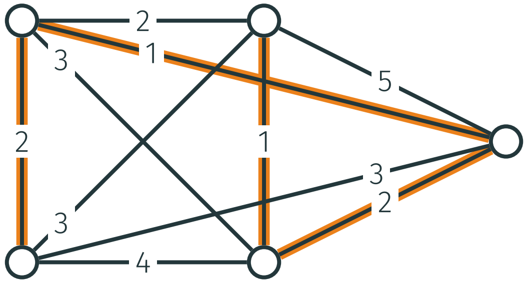
Class P

Definition

P is the class of all search problems that can be solved in polynomial time.

Traveling Salesman Problem

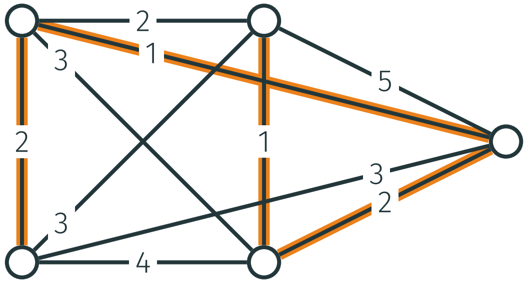
Given a complete weighted graph, find a path of minimum total weight (length) visiting each node exactly once



length: 6

Traveling Salesman Problem

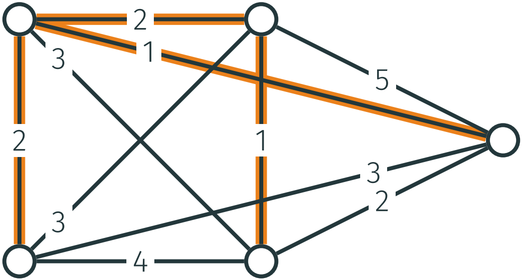
Given a complete weighted graph and a budget b, find a path of total weight (length) $\leq b$ visiting each node exactly once



length: $6 \leq b$

Minimum Spanning Tree

Given a complete weighted graph and a budget b, connect all vertices by $n-1$ edges of minimum total weight (length)



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TSP and MST

MST

Given n cities, connect them by $(n-1)$ roads of minimal total length

TSP and MST

MST

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Can be solved efficiently

TSP and MST

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Given n cities, connect them by $(n-1)$ roads of minimal total length

TSP

Given n cities, connect them in a path of minimal total length

Can be solved efficiently

TSP and MST

MST	TSP
Given n cities, connect them by $(n-1)$ roads of minimal total length	Given n cities, connect them in a path of minimal total length
Can be solved efficiently	No polynomial algorithm known!

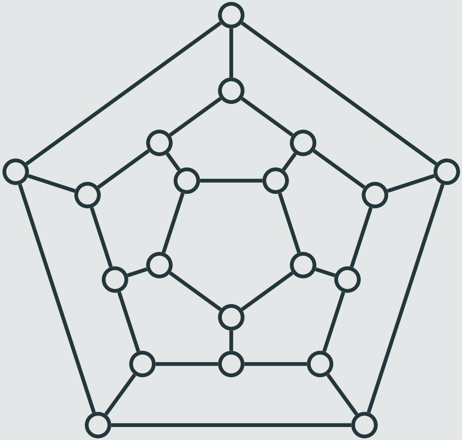
Longest Path

Longest path

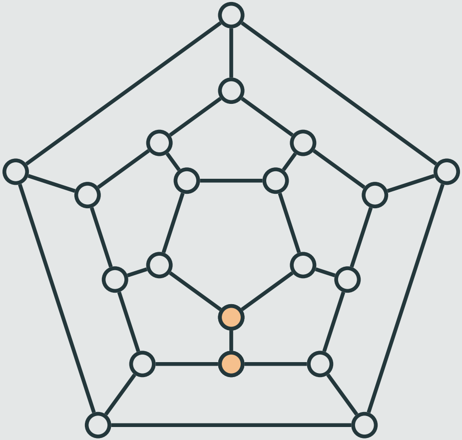
Input: A weighted graph, two vertices s, t, and a budget b.

Output: A simple path (containing no repeated vertices) of total length at least b.

Example



Example



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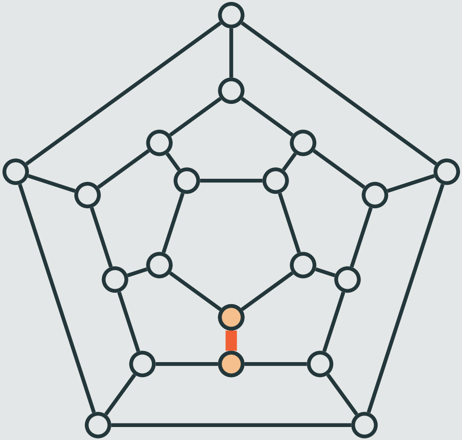

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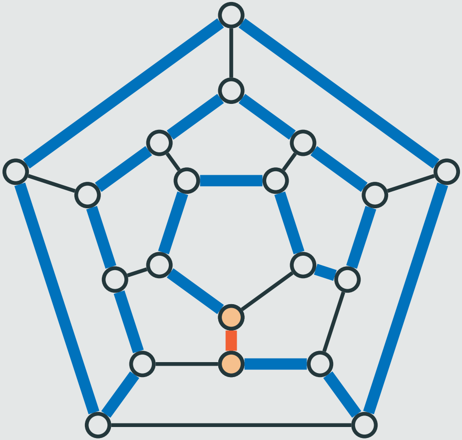

Shortest path

Find a simple path from s to t of total length at most b

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Shortest path	Longest path
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Can be solved efficiently	

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Can be solved efficiently	No polynomial algorithm known!

Integer Linear Programming Problem

Integer linear programming

Input: A set of linear inequalities $\mathbf{A}\mathbf{x} \leq \mathbf{b}$.

Output: Integer solution.

Example

$$
\begin{aligned}
x_1 &\geq 0.5 \\
-x_1 + 8x_2 &\geq 0 \\
-x_1 - 8x_2 &\geq -8
\end{aligned}
$$

Example

$$x_1 \geq 0.5$$
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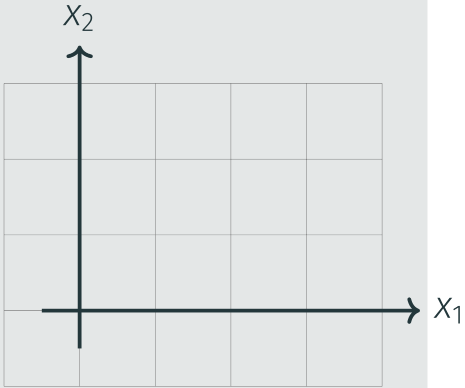

Example

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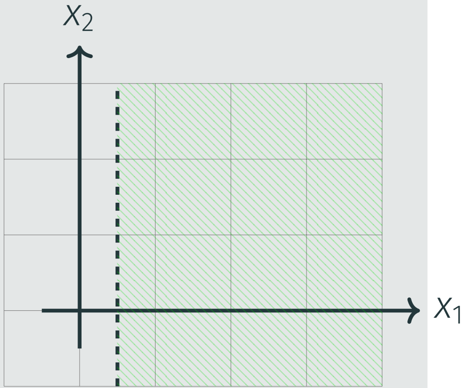

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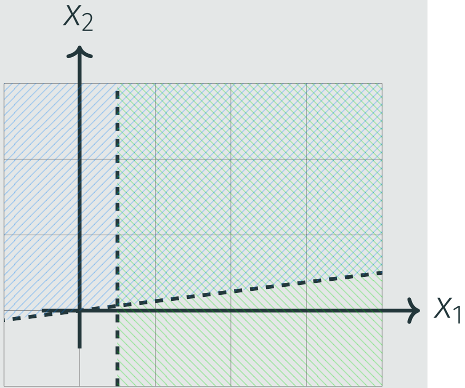

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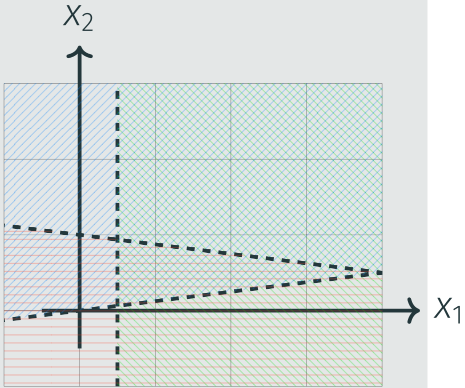

Integer Linear Programming

LP

Find a real
solution of a system of
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Integer Linear Programming

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Integer Linear Programming

LP	ILP
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Integer Linear Programming

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Can be solved efficiently	No polynomial algorithm known!

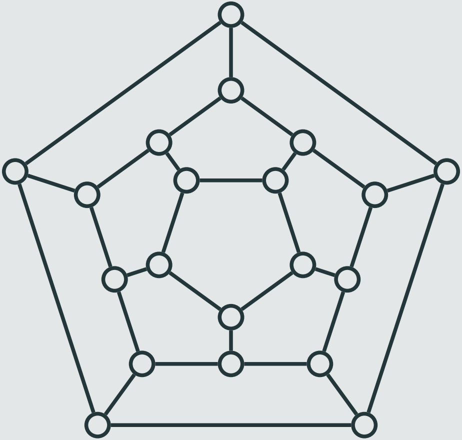
Independent Set Problem

Independent set

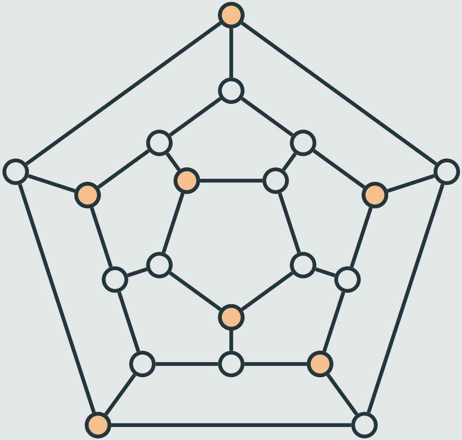
Input: A graph and a budget b.

Output: A subset of vertices of size at least b such that no two of them are adjacent.

Example

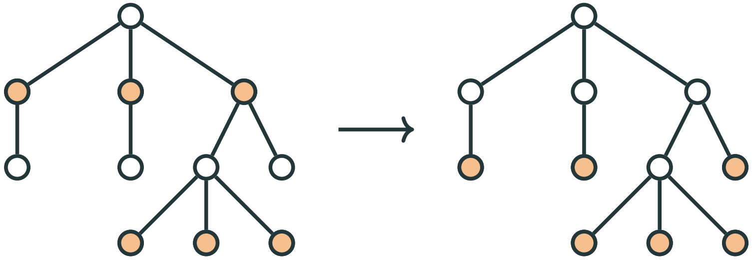


Example



Independent Sets in a Tree

A maximum independent set in a tree can be found by a simple greedy algorithm: it is safe to take into a solution all the leaves.



Independent set in a tree

Find an independent set of size at least b in a given tree

Independent set in a tree

Find an independent set of size at least *b* in a given tree

Can be solved efficiently

Independent set in a tree

Find an independent set of size at least *b* in a given tree

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Independent set in a graph

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Independent set in a tree	Independent set in a graph
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Can be solved efficiently	No polynomial algorithm known!

NP

It turns out that all these hard problems are in a sense a single hard problem: a polynomial time algorithm for any of these problems can be used to solve all of them in polynomial time!

Class P

Problems whose solution can be found efficiently

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Problems whose solution can be found efficiently

- MST
- Shortest path
- LP
- IS on trees

Class P

Problems whose solution can be found efficiently

Class NP

Problems whose solution can be verified efficiently

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Class P

Problems whose solution can be found efficiently

- MST
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Class NP

Problems whose solution can be verified efficiently

- TSP
- Longest path
- ILP
- IS on graphs

The main open problem in Computer Science

Is P equal to NP?

The main open problem in Computer Science

Is P equal to NP?

Millenium Prize Problem

Clay Mathematics Institute: $1M prize for solving the problem

- If P=NP, then all search problems can be solved in polynomial time.

- If P$=$NP, then all search problems can be solved in polynomial time.
- If P$\neq$NP, then there exist search problems that cannot be solved in polynomial time.

Reductions

Informally

We say that a search problem A is reduced to a search problem B and write $A \to B$, if a polynomial time algorithm for B can be used (as a black box) to solve A in polynomial time.

Reduction: $A \to B$

instance I of A

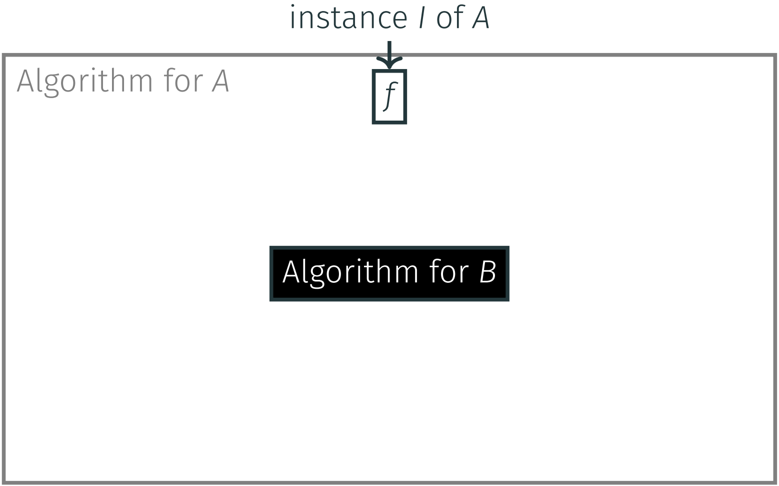
Reduction: $A \to B$

instance I of A

Algorithm for A

Algorithm for B

Reduction: $A \rightarrow B$



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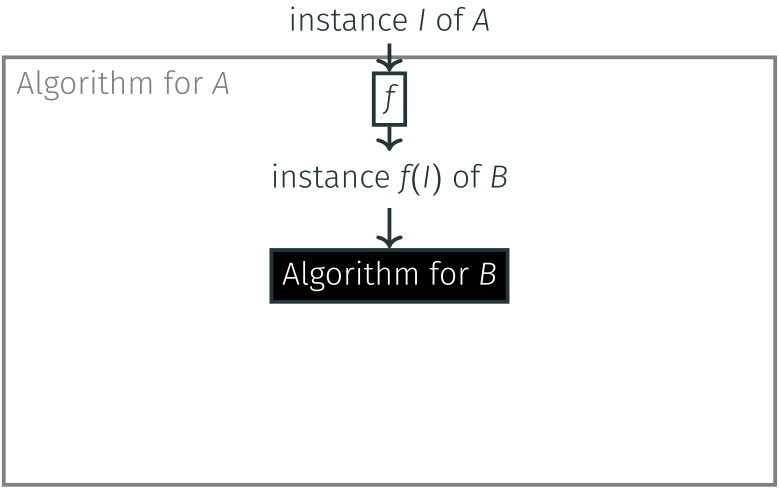

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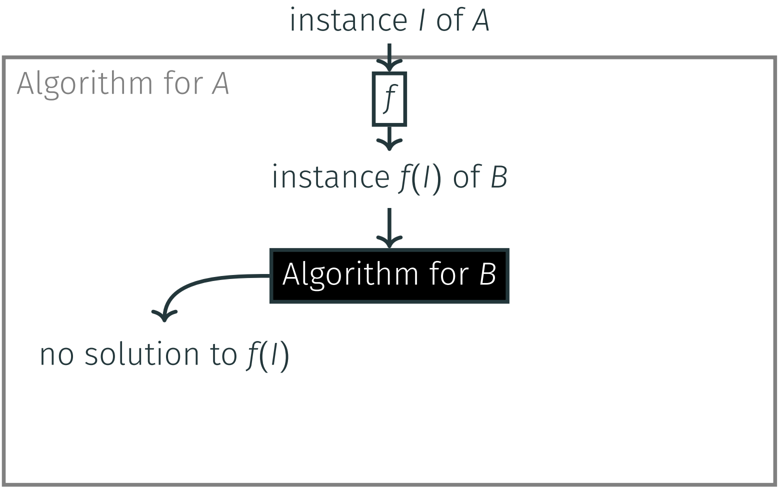

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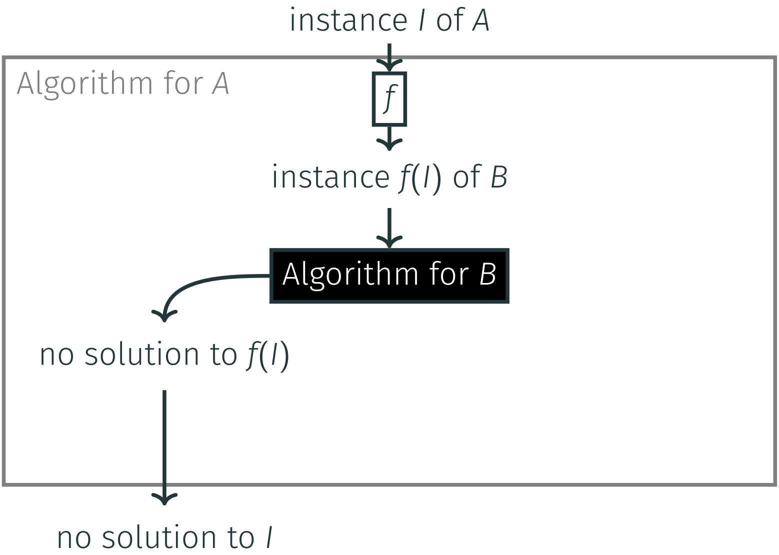

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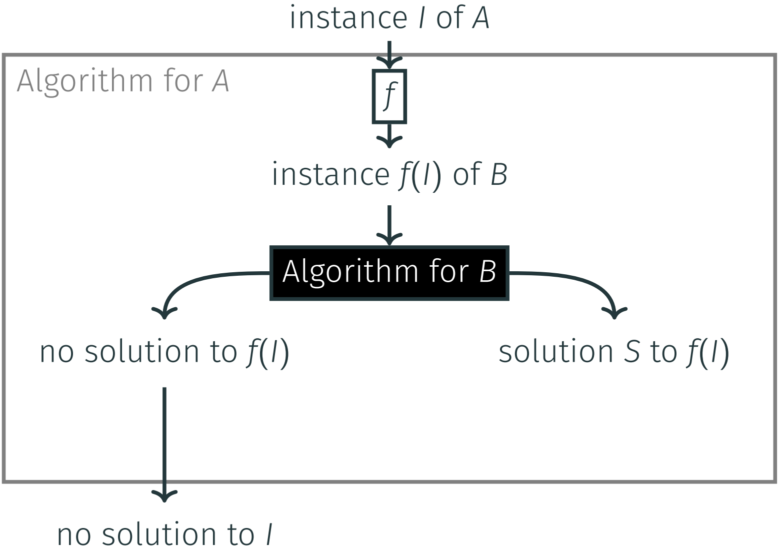

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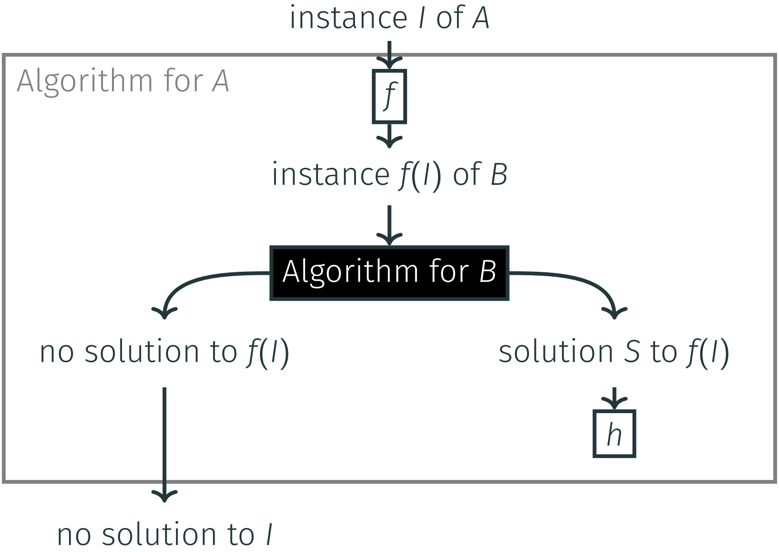

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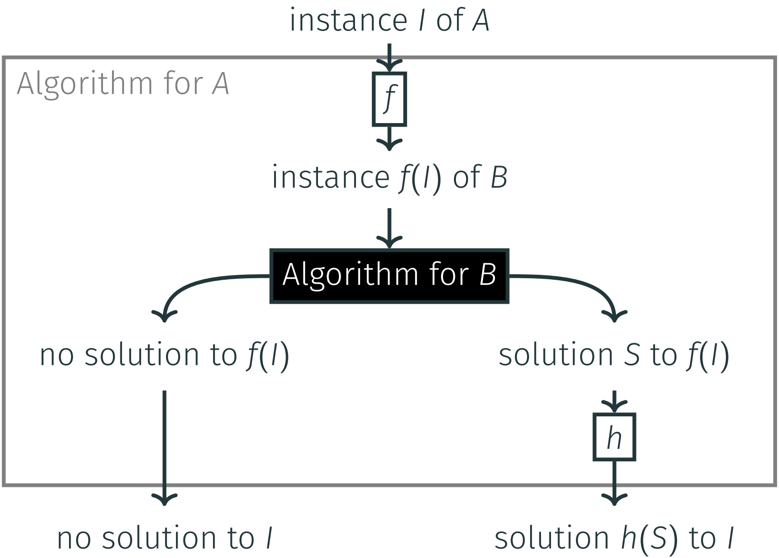
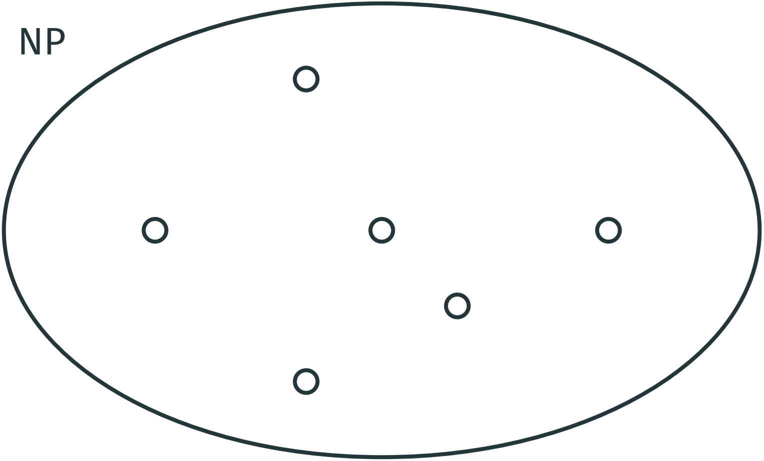


Formally

Definition

We say that a search problem A is reduced to a search problem B and write $A \to B$, if there exists a polynomial time algorithm f that converts any instance I of A into an instance $f(I)$ of B, together with a polynomial time algorithm h that converts any solution S to $f(I)$ back to a solution $h(S)$ to A. If there is no solution to $f(I)$, then there is no solution to I.

Graph of Search Problems



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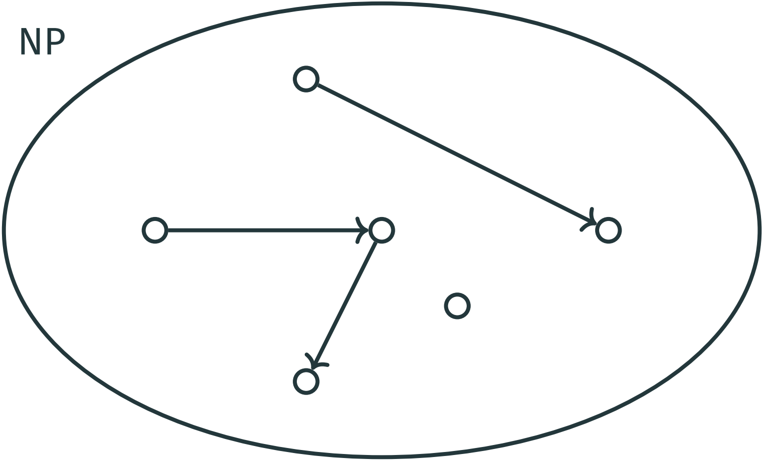

NP-complete Problems

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A search problem is called NP-complete if all other search problems reduce to it.

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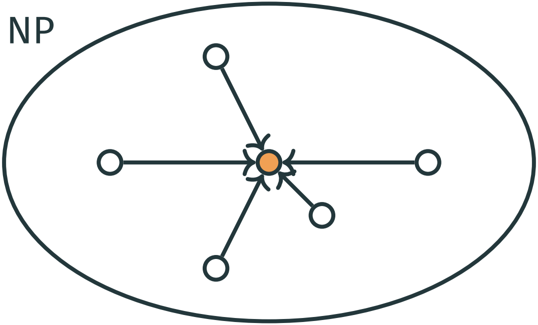


Do they exist?

It's not at all immediate that NP-complete problems even exist. We'll see later that all hard problems that we've seen in the previous part are in fact NP-complete!

Two ways of using $A \to B$:

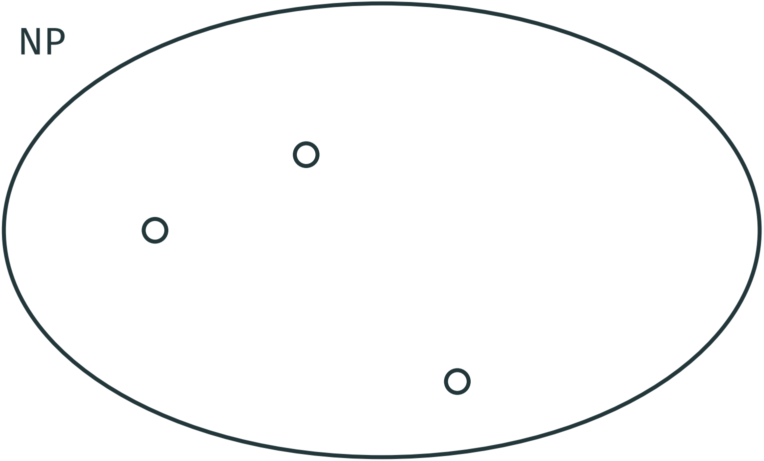
- if B is easy (can be solved in polynomial time), then so is A
- if A is hard (cannot be solved in polynomial time), then so is B

Reductions Compose

Lemma

If $A \rightarrow B$ and $B \rightarrow C$, then $A \rightarrow C$.

Pictorially



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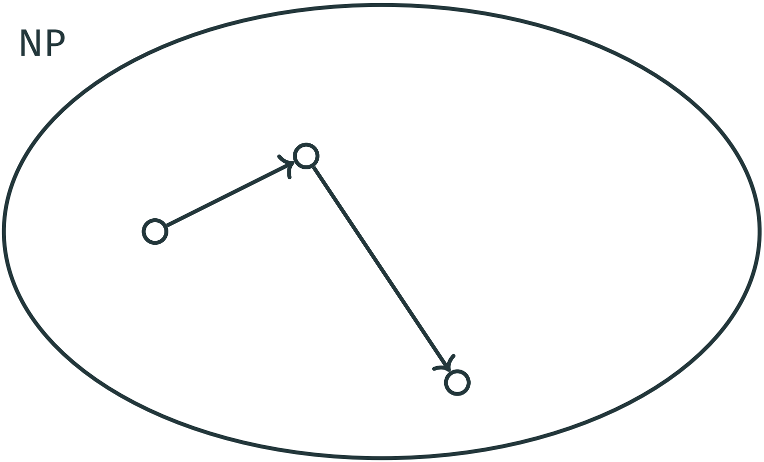

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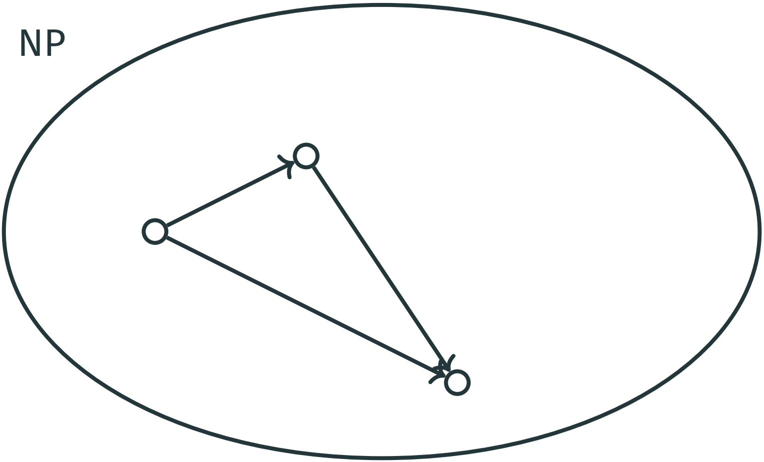

Showing NP-completeness

Corollary

If $A \to B$ and A is NP-complete, then so is B.

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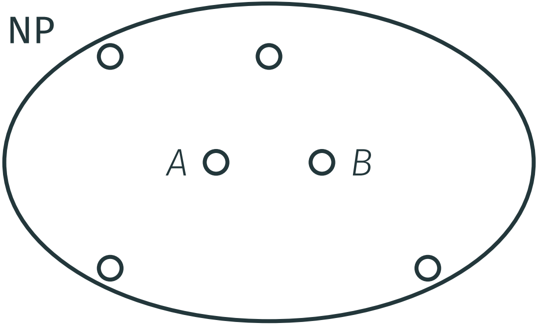
If $A \to B$ and A is **NP**-complete, then so is B.



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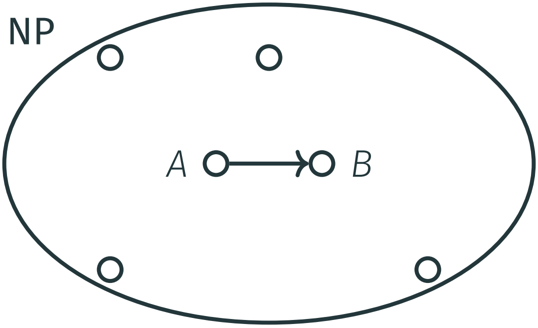

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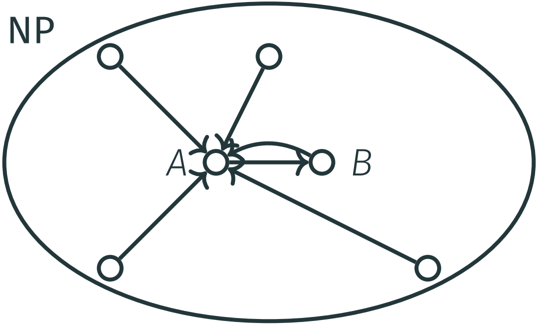

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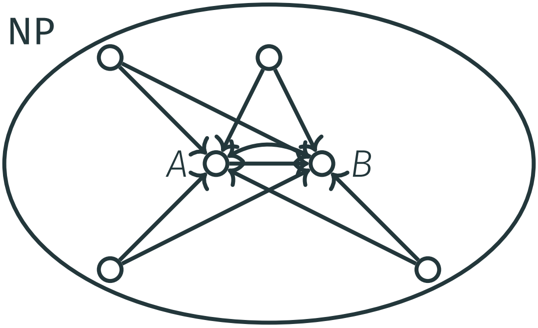
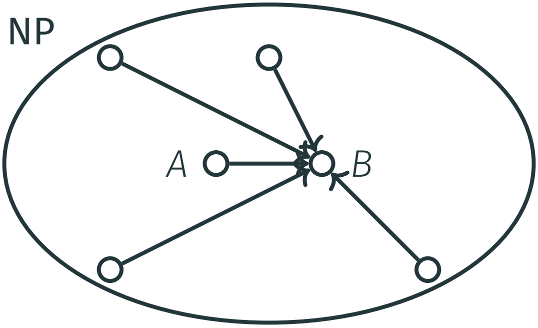


Showing NP-completeness

Corollary

If $A \rightarrow B$ and A is NP-complete, then so is B.



NP-Completeness of SAT

Goal

Show that every search problem reduces to SAT.

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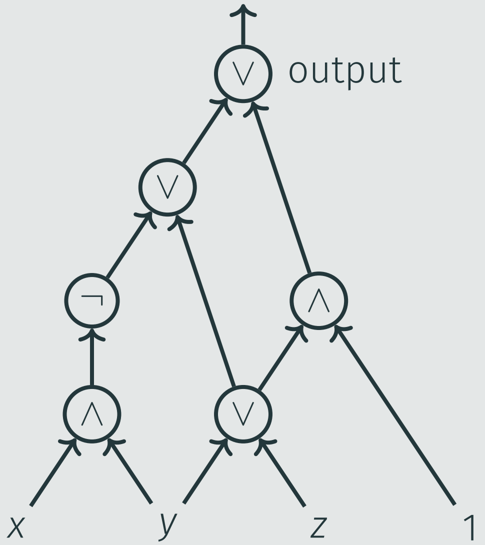
Instead, we show that any problem reduces to Circuit SAT problem, which, in turn, reduces to SAT.

Circuit



Definition

A circuit is a directed acyclic graph of in-degree at most 2. Nodes of in-degree 0 are called inputs and are marked by Boolean variables and constants. Nodes of in-degree 1 and 2 are called gates: gates of in-degree 1 are labeled with NOT, gates of in-degree 2 are labeled with AND or OR. One of the sinks is marked as output.

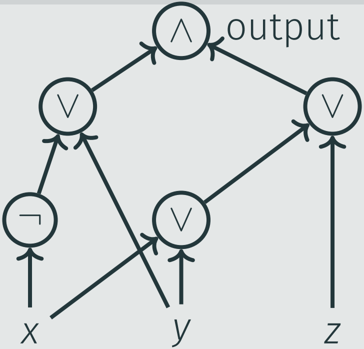
Circuit-SAT

Input: A circuit.

Output: An assignment of Boolean values to the input variables of the circuit that makes the output true.

SAT is a special case of Circuit-SAT as a SAT formula can be represented as a circuit:

Example: $(x \vee y \vee z)(y \vee \bar{x})$



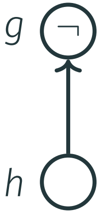
Circuit-SAT $\rightarrow$ SAT

To reduce Circuit-SAT to SAT, we need to design a polynomial time algorithm that for a given circuit outputs a SAT formula which is satisfiable, if and only if the circuit is satisfiable

Idea

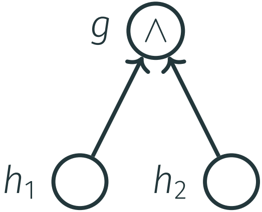
- Introduce a Boolean variable for each gate
- For each gate, write down a few clauses that describe the relationship between this gate and its direct predecessors

NOT Gates



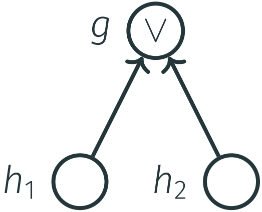
$$(h \vee g)(\overline{h} \vee \overline{g})$$

AND Gates



$$(h_1 \vee \overline{g})(h_2 \vee \overline{g})(\overline{h}_1 \vee \overline{h}_2 \vee g)$$

OR Gates



$$(\overline{h}_1 \vee g)(\overline{h}_2 \vee g)(h_1 \vee h_2 \vee \overline{g})$$

Output Gate

g output (g)

- The resulting SAT formula is consistent with the initial circuit: in any satisfying assignment of the formula, the value of g is equal to the value of the gate labeled with g in the circuit

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- Therefore, the SAT formula and the circuit are equisatisfiable
- The reduction takes polynomial time

Goal

Reduce every search problem to Circuit-SAT.

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Goal

Reduce every search problem to Circuit-SAT.

- Let A be a search problem
- We know that there exists an algorithm $\mathcal{C}$ that takes an instance I of A and a candidate solution S and checks whether S is a solution for I in time polynomial in $|I|$
- In particular, $|S|$ is polynomial in $|I|$

Turn an Algorithm into a Circuit

- Note that a computer is in fact a circuit implemented on a chip

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- Each step of the algorithm $\mathcal{C}(I, S)$ is performed by this computer's circuit
- This gives a circuit of size polynomial in $|I|$ that has input bits for I and S and outputs whether S is a solution for I (a separate circuit for each input length)

Reduction

To solve an instance I of the problem A:

- take a circuit corresponding to $\mathcal{C}(I, \cdot)$

Reduction

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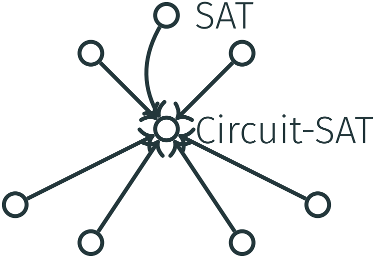
- the inputs to this circuit encode candidate solutions

Reduction

To solve an instance I of the problem A:

- take a circuit corresponding to $\mathcal{C}(I, \cdot)$
- the inputs to this circuit encode candidate solutions
- use a Circuit-SAT algorithm for this circuit to find a solution (if exists)

Summary



Summary