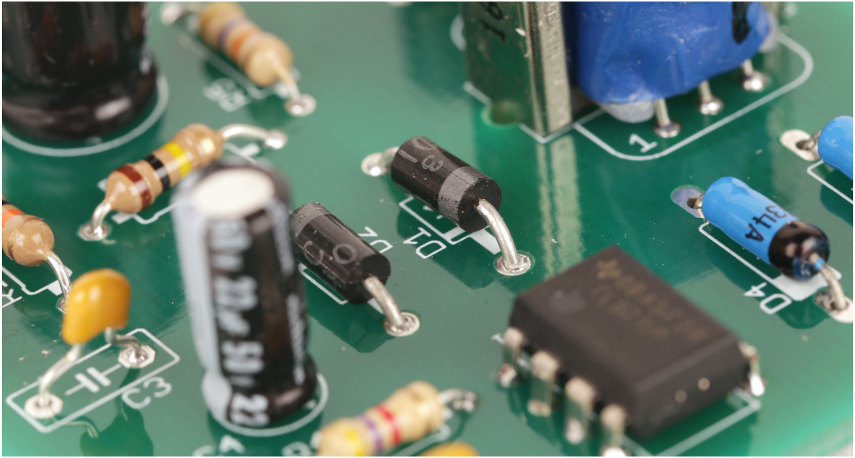
Gems of TCS

Circuit Complexity II

Sasha Golovnev

October 27, 2021


C3
D1
D2
D4

Boolean Circuits

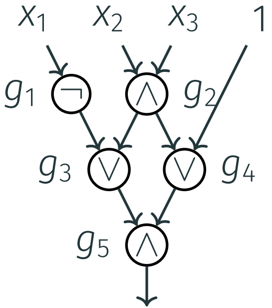
$$f\colon \{0,1\}^n \to \{0,1\}$$

$$\begin{aligned}
g_1 &= \neg x_1 \\
g_2 &= x_2 \wedge x_3 \\
g_3 &= g_1 \vee g_2 \\
g_4 &= g_2 \vee 1 \\
g_5 &= g_3 \wedge g_4
\end{aligned}$$

Boolean Circuits

$$f\colon \{0,1\}^n \to \{0,1\}$$

$$\begin{aligned} g_1 &= \neg x_1 \\ g_2 &= x_2 \wedge x_3 \\ g_3 &= g_1 \vee g_2 \\ g_4 &= g_2 \vee 1 \\ g_5 &= g_3 \wedge g_4 \end{aligned}$$



Exponential Bounds

Lower Bound [Sha1949]

Almost all functions of n variables have circuit size

$$\geq 2^n/n$$

Exponential Bounds

Lower Bound [Sha1949]

Almost all functions of n variables have circuit size

$$\geq 2^n/n$$

Upper Bound [Lup1958]

Any function can be computed by a circuit of size

$$\leq 2^n/n$$

Symmetric Functions

$f\colon \{0,1\}^n \to \{0,1\}$ is symmetric if it depends only on the number of ones in the input, and not on positions of these ones.

Symmetric Functions

$f\colon \{0,1\}^n \to \{0,1\}$ is symmetric if it depends only on the number of ones in the input, and not on positions of these ones.

- $\text{And}(x) = 1$ iff $x_1 = \ldots = x_n = 1$

Symmetric Functions

$f\colon \{0,1\}^n \to \{0,1\}$ is symmetric if it depends only on the number of ones in the input, and not on positions of these ones.

- $\text{And}(x) = 1$ iff $x_1 = \ldots = x_n = 1$
- $\text{Or}(x) = 1$ iff $x_1 + \ldots + x_n \geq 1$

Symmetric Functions

$f\colon \{0,1\}^n \to \{0,1\}$ is symmetric if it depends only on the number of ones in the input, and not on positions of these ones.

- $\text{And}(x) = 1$ iff $x_1 = \ldots = x_n = 1$
- $\text{Or}(x) = 1$ iff $x_1 + \ldots + x_n \geq 1$
- $\bigoplus(x) = 1$ iff $x_1 + \ldots + x_n \equiv 1 \mod 2$

Symmetric Functions

$f\colon \{0,1\}^n \to \{0,1\}$ is symmetric if it depends only on the number of ones in the input, and not on positions of these ones.

- $\mathrm{And}(x) = 1$ iff $x_1 = \ldots = x_n = 1$
- $\mathrm{Or}(x) = 1$ iff $x_1 + \ldots + x_n \geq 1$
- $\bigoplus(x) = 1$ iff $x_1 + \ldots + x_n \equiv 1 \mod 2$
- $\mathrm{Mod}_k(x) = 1$ iff $x_1 + \ldots + x_n \equiv 0 \mod k$

Symmetric Functions

$f\colon \{0,1\}^n \to \{0,1\}$ is symmetric if it depends only on the number of ones in the input, and not on positions of these ones.

- $\mathrm{And}(x) = 1$ iff $x_1 = \ldots = x_n = 1$
- $\mathrm{Or}(x) = 1$ iff $x_1 + \ldots + x_n \geq 1$
- $\bigoplus(x) = 1$ iff $x_1 + \ldots + x_n \equiv 1 \mod 2$
- $\mathrm{Mod}_k(x) = 1$ iff $x_1 + \ldots + x_n \equiv 0 \mod k$
- $\mathrm{Maj}(x) = 1$ iff $x_1 + \ldots + x_n \geq n/2$

Symmetric Functions

$f\colon \{0,1\}^n \to \{0,1\}$ is symmetric if it depends only on the number of ones in the input, and not on positions of these ones.

- $\mathrm{And}(x) = 1$ iff $x_1 = \ldots = x_n = 1$
- $\mathrm{Or}(x) = 1$ iff $x_1 + \ldots + x_n \geq 1$
- $\bigoplus(x) = 1$ iff $x_1 + \ldots + x_n \equiv 1 \mod 2$
- $\mathrm{Mod}_k(x) = 1$ iff $x_1 + \ldots + x_n \equiv 0 \mod k$
- $\mathrm{Maj}(x) = 1$ iff $x_1 + \ldots + x_n \geq n/2$
- $\mathrm{Th}_k(x) = 1$ iff $x_1 + \ldots + x_n \geq k$

Symmetric Functions. Equiv Def

$f\colon \{0,1\}^n \to \{0,1\}$ is symmetric iff

$$f = g(x_1 + \ldots + x_n)$$

for some $g\colon \{0,\ldots,n\} \to \{0,1\}$.

Symmetric Functions. Equiv Def

$f\colon \{0,1\}^n \to \{0,1\}$ is symmetric iff

$$f = g(x_1 + \ldots + x_n)$$

for some $g\colon \{0,\ldots,n\} \to \{0,1\}$.

$f\colon \{0,1\}^n \to \{0,1\}$ is symmetric iff

$$f = h(\mathrm{Sum}_n(x_1,\ldots,x_n))$$

for some $h\colon \{0,1\}^{\log n} \to \{0,1\}$, where $\mathrm{Sum}_n\colon \{0,1\}^n \to \{0,1\}^{\log n}$.

Complexity of Sum

$\mathrm{Sum}_3(x_1, x_2, x_3) \in \{0, 1\}^2$

Complexity of Sum

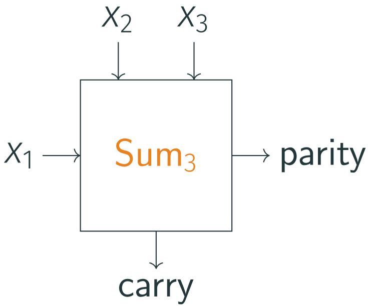
$\mathrm{Sum}_3(x_1, x_2, x_3) \in \{0, 1\}^2$

$\mathrm{Sum}_3(x_1, x_2, x_3) = (\mathrm{carry}, \mathrm{parity})$

Complexity of Sum

$\mathrm{Sum}_3(x_1, x_2, x_3) \in \{0, 1\}^2$

$\mathrm{Sum}_3(x_1, x_2, x_3) = (\mathrm{carry}, \mathrm{parity})$

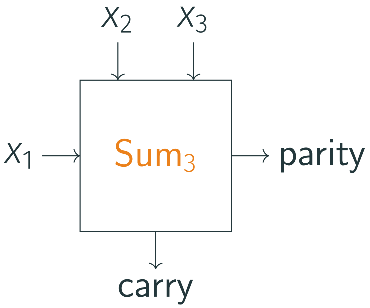


Complexity of Sum

$$\mathrm{Sum}_3(x_1, x_2, x_3) \in \{0, 1\}^2$$

$$\mathrm{Sum}_3(x_1, x_2, x_3) = (\mathrm{carry}, \mathrm{parity})$$

$$\mathrm{Size}(\mathrm{Sum}_3) = O(1)$$

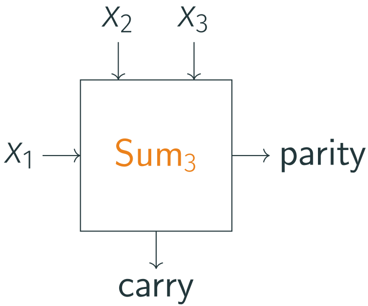


Complexity of Sum

$$\mathrm{Sum}_3(x_1, x_2, x_3) \in \{0,1\}^2$$

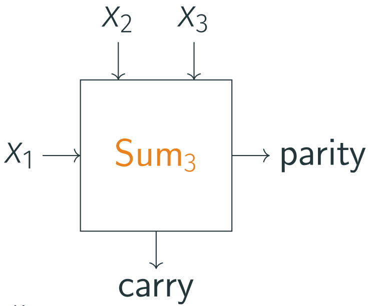
$$\mathrm{Sum}_3(x_1, x_2, x_3) = (\mathrm{carry}, \mathrm{parity})$$

$$\mathrm{Size}(\mathrm{Sum}_3) = O(1)$$



Sum_5?

Complexity of Sum

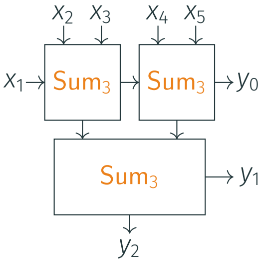


$$\mathrm{Sum}_3(x_1, x_2, x_3) \in \{0,1\}^2$$

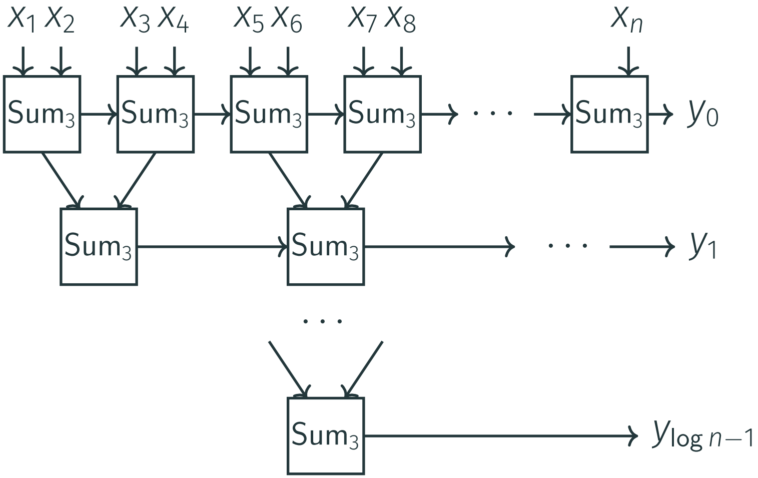
$$\mathrm{Sum}_3(x_1, x_2, x_3) = (\mathrm{carry}, \mathrm{parity})$$

$$\mathrm{Size}(\mathrm{Sum}_3) = O(1)$$

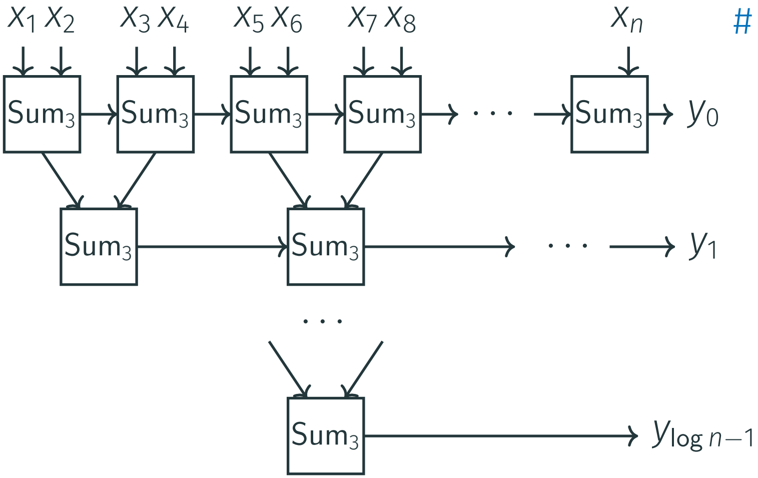
Sum$_5$?



Complexity of Sum_n

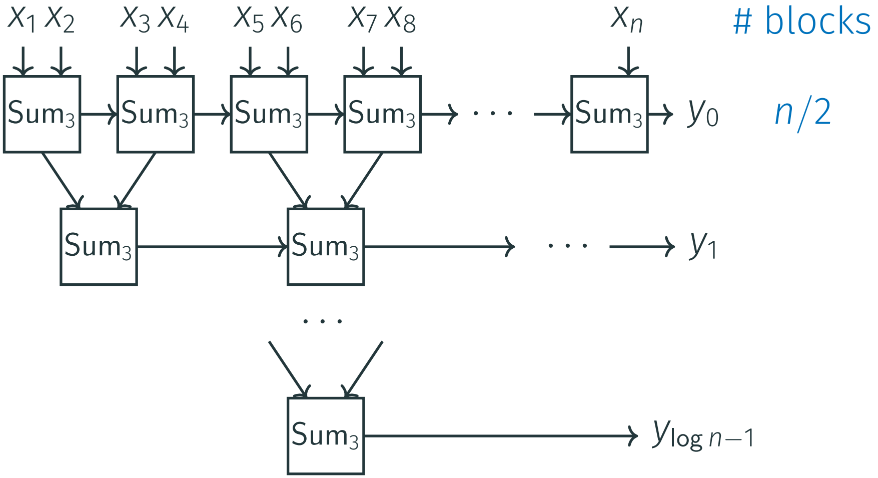


Complexity of Sum_n

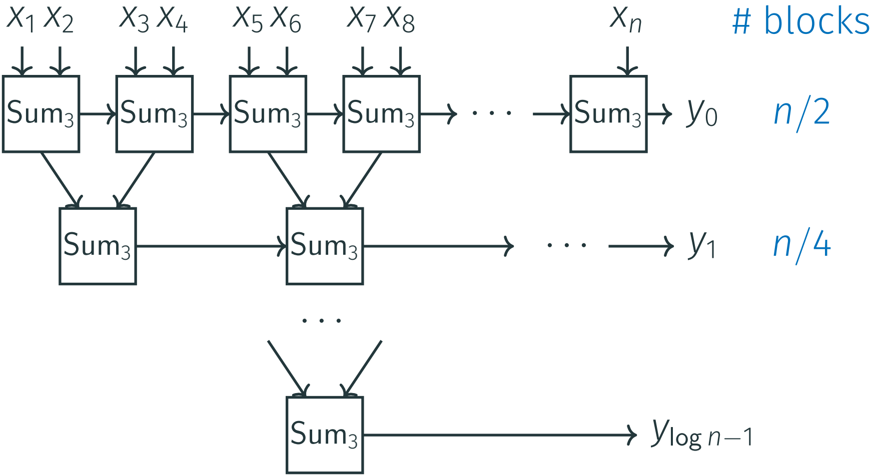


blocks

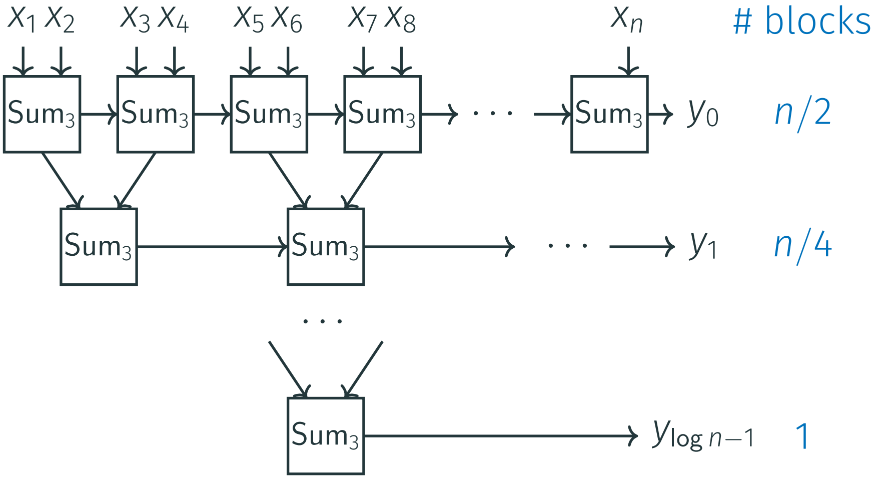
Complexity of Sum_n



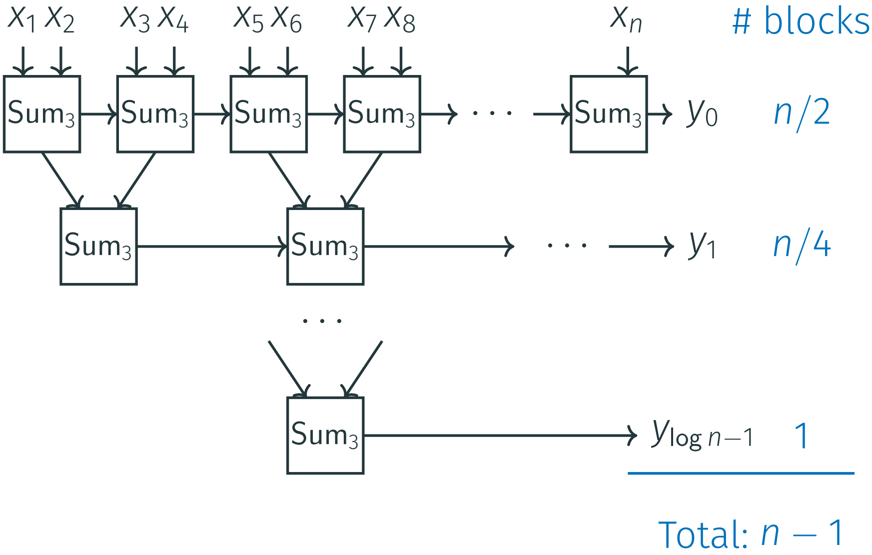
Complexity of Sum_n



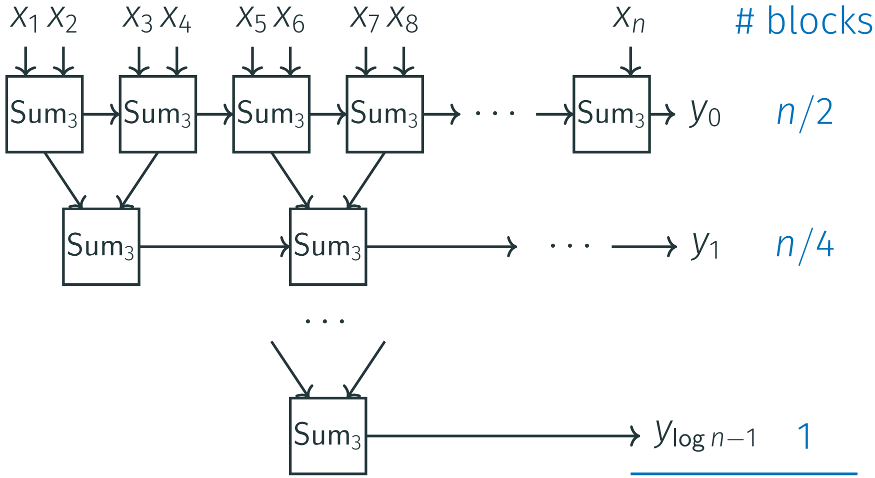
Complexity of Sum_n



Complexity of Sum_n



Complexity of Sum_n



$\mathrm{Size}(\mathrm{Sum}_n) < n \cdot \mathrm{Size}(\mathrm{Sum}_3) = O(n)$

Complexity of Symmetric Functions

Theorem

If $f\colon \{0,1\}^n \to \{0,1\}$ is a symmetric function, then

$$\mathsf{Size}(f) \leq O(n)\,.$$

Complexity of Symmetric Functions

Theorem

If $f\colon \{0,1\}^n \to \{0,1\}$ is a symmetric function, then

$$\mathrm{Size}(f) \leq O(n)\,.$$

$$f = h(\mathrm{Sum}_n(x_1, \ldots, x_n)),\;\; h\colon \{0,1\}^{\log n} \to \{0,1\}$$

Complexity of Symmetric Functions

Theorem

If $f\colon \{0,1\}^n \to \{0,1\}$ is a symmetric function, then

$$\mathsf{Size}(f) \leq O(n)\,.$$

$$f = h(\mathsf{Sum}_n(x_1, \ldots, x_n)),\ \ h\colon \{0,1\}^{\log n} \to \{0,1\}$$

$$\mathsf{Size}(f) \leq \mathsf{Size}(\mathsf{Sum}_n) + \mathsf{Size}(h)$$

Complexity of Symmetric Functions

Theorem

If $f\colon \{0,1\}^n \to \{0,1\}$ is a symmetric function, then

$$\mathrm{Size}(f) \le O(n)\,.$$

$$f = h(\mathrm{Sum}_n(x_1,\ldots,x_n)),\ \ h\colon \{0,1\}^{\log n} \to \{0,1\}$$

$$\mathrm{Size}(f) \le \mathrm{Size}(\mathrm{Sum}_n) + \mathrm{Size}(h)$$

$$\mathrm{Size}(\mathrm{Sum}_n) = O(n)$$

Complexity of Symmetric Functions

Theorem

If $f\colon \{0,1\}^n \to \{0,1\}$ is a symmetric function, then

$$\mathrm{Size}(f) \leq O(n)\,.$$

$$f = h(\mathrm{Sum}_n(x_1, \ldots, x_n)),\ \ h\colon \{0,1\}^{\log n} \to \{0,1\}$$

$$\mathrm{Size}(f) \leq \mathrm{Size}(\mathrm{Sum}_n) + \mathrm{Size}(h)$$

$$\mathrm{Size}(\mathrm{Sum}_n) = O(n)$$

$$\mathrm{Size}(h) \leq 10 \cdot 2^{\log n} = O(n)$$

Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

- $k = 1, \mathrm{Th}_1 = \mathrm{Or}$

Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

- $k = 1, \mathrm{Th}_1 = \mathrm{Or}, \mathrm{Size}(\mathrm{Th}_1) = n - 1$

Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

- $k = 1, \mathrm{Th}_1 = \mathrm{Or}, \mathrm{Size}(\mathrm{Th}_1) = n - 1$
- $k = 2$

Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

- $k = 1, \mathrm{Th}_1 = \mathrm{Or}, \mathrm{Size}(\mathrm{Th}_1) = n - 1$
- $k = 2, \mathrm{Size}(\mathrm{Th}_2) = O(n)$

Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

- $k = 1, \mathrm{Th}_1 = \mathrm{Or}, \mathrm{Size}(\mathrm{Th}_1) = n - 1$
- $k = 2, \mathrm{Size}(\mathrm{Th}_2) = O(n)$
- $\mathrm{Size}(\mathrm{Th}_2) < 3n$

Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

- $k = 1, \mathrm{Th}_1 = \mathrm{Or}, \mathrm{Size}(\mathrm{Th}_1) = n - 1$
- $k = 2, \mathrm{Size}(\mathrm{Th}_2) = O(n)$
- $\mathrm{Size}(\mathrm{Th}_2) < 3n$
- Two rounds of "Bubble Sort"

Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

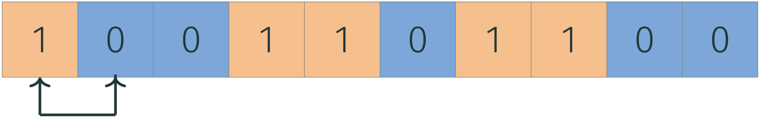
- $k = 1, \mathrm{Th}_1 = \mathrm{Or}, \mathrm{Size}(\mathrm{Th}_1) = n - 1$
- $k = 2, \mathrm{Size}(\mathrm{Th}_2) = O(n)$
- $\mathrm{Size}(\mathrm{Th}_2) < 3n$
- Two rounds of “Bubble Sort”

1	0	0	1	1	0	1	1	0	0

Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

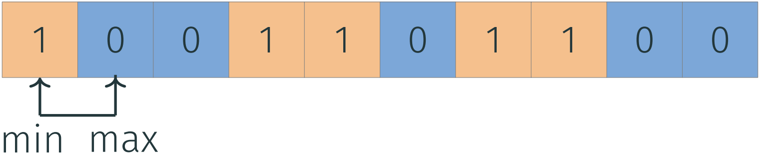
- $k = 1, \mathrm{Th}_1 = \mathrm{Or}, \mathrm{Size}(\mathrm{Th}_1) = n - 1$
- $k = 2, \mathrm{Size}(\mathrm{Th}_2) = O(n)$
- $\mathrm{Size}(\mathrm{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

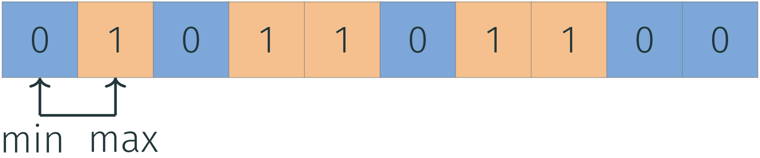
- $k = 1, \mathrm{Th}_1 = \mathrm{Or}, \mathrm{Size}(\mathrm{Th}_1) = n - 1$
- $k = 2, \mathrm{Size}(\mathrm{Th}_2) = O(n)$
- $\mathrm{Size}(\mathrm{Th}_2) < 3n$
- Two rounds of "Bubble Sort"



Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

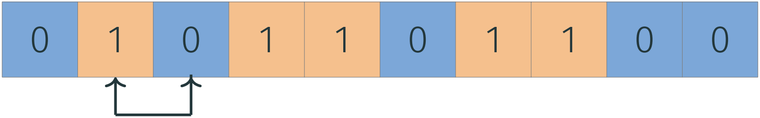
- $k = 1, \mathrm{Th}_1 = \mathrm{Or}, \mathrm{Size}(\mathrm{Th}_1) = n - 1$
- $k = 2, \mathrm{Size}(\mathrm{Th}_2) = O(n)$
- $\mathrm{Size}(\mathrm{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

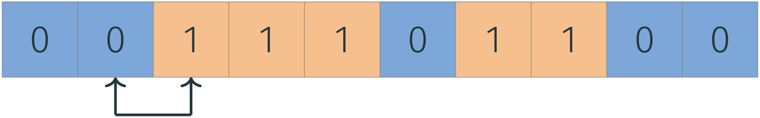
- $k = 1$, $\mathrm{Th}_1 = \mathrm{Or}$, $\mathrm{Size}(\mathrm{Th}_1) = n - 1$
- $k = 2$, $\mathrm{Size}(\mathrm{Th}_2) = O(n)$
- $\mathrm{Size}(\mathrm{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

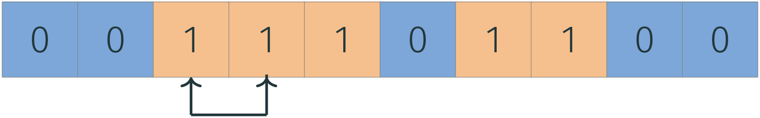
- $k = 1, \mathrm{Th}_1 = \mathrm{Or}, \mathrm{Size}(\mathrm{Th}_1) = n - 1$
- $k = 2, \mathrm{Size}(\mathrm{Th}_2) = O(n)$
- $\mathrm{Size}(\mathrm{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

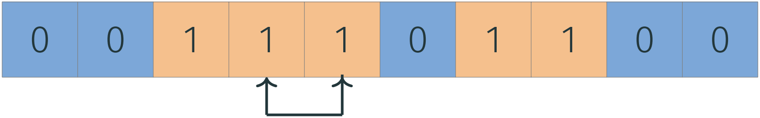
- $k = 1, \mathrm{Th}_1 = \mathrm{Or}, \mathrm{Size}(\mathrm{Th}_1) = n - 1$
- $k = 2, \mathrm{Size}(\mathrm{Th}_2) = O(n)$
- $\mathrm{Size}(\mathrm{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

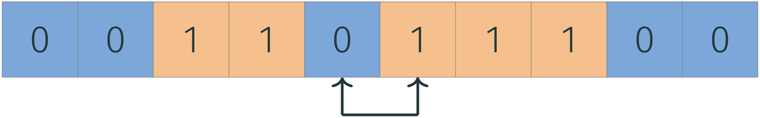
- $k = 1, \mathrm{Th}_1 = \mathrm{Or}, \mathrm{Size}(\mathrm{Th}_1) = n - 1$
- $k = 2, \mathrm{Size}(\mathrm{Th}_2) = O(n)$
- $\mathrm{Size}(\mathrm{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

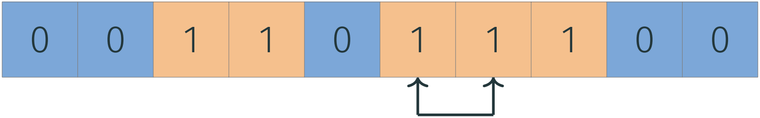
- $k = 1, \mathrm{Th}_1 = \mathrm{Or}, \mathrm{Size}(\mathrm{Th}_1) = n - 1$
- $k = 2, \mathrm{Size}(\mathrm{Th}_2) = O(n)$
- $\mathrm{Size}(\mathrm{Th}_2) < 3n$
- Two rounds of "Bubble Sort"



Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

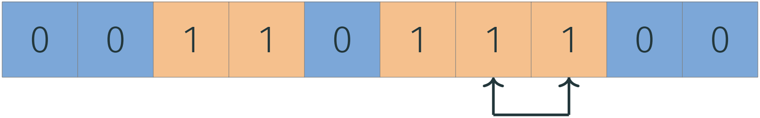
- $k = 1, \mathrm{Th}_1 = \mathrm{Or}, \mathrm{Size}(\mathrm{Th}_1) = n - 1$
- $k = 2, \mathrm{Size}(\mathrm{Th}_2) = O(n)$
- $\mathrm{Size}(\mathrm{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

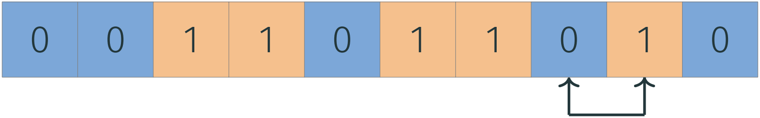
- $k = 1, \mathrm{Th}_1 = \mathrm{Or}, \mathrm{Size}(\mathrm{Th}_1) = n - 1$
- $k = 2, \mathrm{Size}(\mathrm{Th}_2) = O(n)$
- $\mathrm{Size}(\mathrm{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

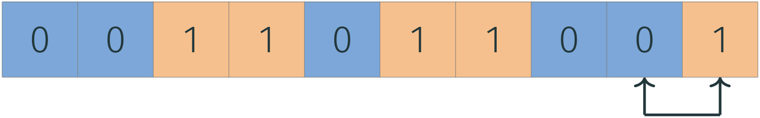
- $k = 1, \mathrm{Th}_1 = \mathrm{Or}, \mathrm{Size}(\mathrm{Th}_1) = n - 1$
- $k = 2, \mathrm{Size}(\mathrm{Th}_2) = O(n)$
- $\mathrm{Size}(\mathrm{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

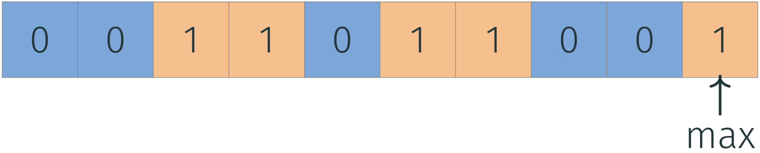
- $k = 1$, $\mathrm{Th}_1 = \mathrm{Or}$, $\mathrm{Size}(\mathrm{Th}_1) = n - 1$
- $k = 2$, $\mathrm{Size}(\mathrm{Th}_2) = O(n)$
- $\mathrm{Size}(\mathrm{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

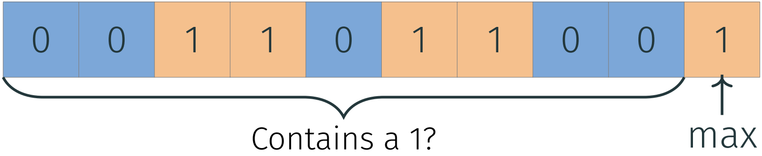
- $k = 1, \mathrm{Th}_1 = \mathrm{Or}, \mathrm{Size}(\mathrm{Th}_1) = n - 1$
- $k = 2, \mathrm{Size}(\mathrm{Th}_2) = O(n)$
- $\mathrm{Size}(\mathrm{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

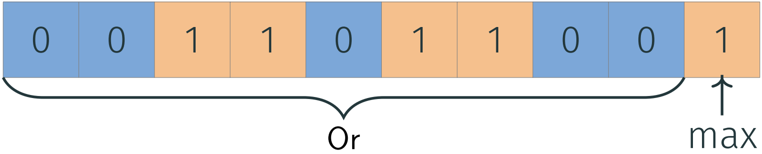
- $k = 1$, $\mathrm{Th}_1 = \mathrm{Or}$, $\mathrm{Size}(\mathrm{Th}_1) = n - 1$
- $k = 2$, $\mathrm{Size}(\mathrm{Th}_2) = O(n)$
- $\mathrm{Size}(\mathrm{Th}_2) < 3n$
- Two rounds of "Bubble Sort"



Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

- $k = 1, \mathrm{Th}_1 = \mathrm{Or}, \mathrm{Size}(\mathrm{Th}_1) = n - 1$
- $k = 2, \mathrm{Size}(\mathrm{Th}_2) = O(n)$
- $\mathrm{Size}(\mathrm{Th}_2) < 3n$
- Two rounds of "Bubble Sort"



Th$_2$. Upper Bound

$$\boxed{\begin{array}{ccc} X_1 & \cdots & X_{\sqrt{n}} \\ \vdots & & \vdots \\ X_{n-\sqrt{n}+1} & & X_n \end{array}}$$

Th_2. Upper Bound

$$\boxed{\begin{array}{ccc} x_1 & \cdots & x_{\sqrt{n}} \\ \vdots & & \vdots \\ x_{n-\sqrt{n}+1} & & x_n \end{array}}$$

$\mathrm{Th}_2(x_1, \ldots, x_n) = 1$ iff

Th_2. Upper Bound

$$\boxed{\begin{array}{ccc} x_1 & \cdots & x_{\sqrt{n}} \\ \vdots & & \vdots \\ x_{n-\sqrt{n}+1} & & x_n \end{array}}$$

$\mathrm{Th}_2(x_1, \ldots, x_n) = 1$ iff

there are two cols with 1s

OR

there are two rows with 1s

Th_2. Upper Bound

$$
\begin{array}{r|ccc|}
\hline
y_1 = \mathrm{Or} & x_1 & \cdots & x_{\sqrt{n}} \\
y_2 = \mathrm{Or} & & & \\
\vdots & \vdots & & \vdots \\
y_{\sqrt{n}} = \mathrm{Or} & x_{n-\sqrt{n}+1} & & x_n \\
\hline
\end{array}
$$

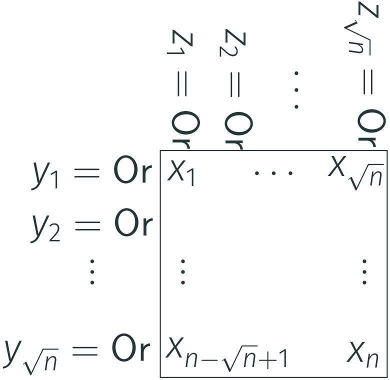
$\mathrm{Th}_2(x_1, \ldots, x_n) = 1$ iff

there are two cols with 1s

OR

there are two rows with 1s

Th$_2$. Upper Bound



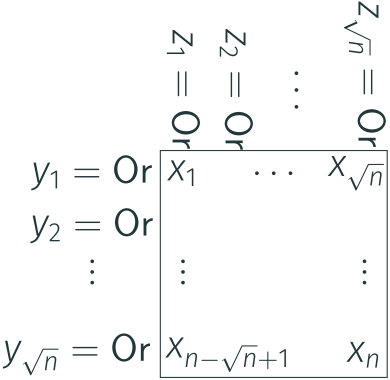
$\text{Th}_2(x_1, \ldots, x_n) = 1$ iff

there are two cols with 1s

OR

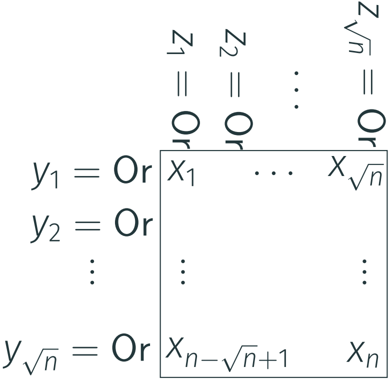
there are two rows with 1s

Th_2. Upper Bound



$$\mathrm{Th}_2(x_1, \ldots, x_n) = \mathrm{Th}_2(y_1, \ldots, y_{\sqrt{n}}) \ \mathrm{Or} \ \mathrm{Th}_2(z_1, \ldots, z_{\sqrt{n}})$$

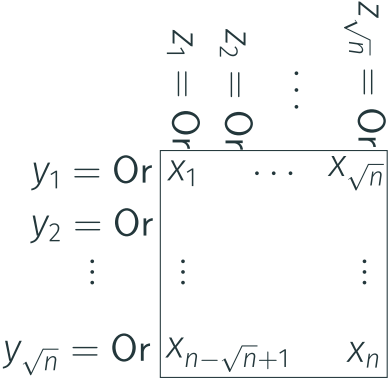
Th_2. Upper Bound



$$\mathrm{Th}_2(x_1, \ldots, x_n) = \mathrm{Th}_2(y_1, \ldots, y_{\sqrt{n}}) \ \text{ Or } \ \mathrm{Th}_2(z_1, \ldots, z_{\sqrt{n}})$$

$$\mathrm{Size}(\mathrm{Th}_2(n)) \leq 2n + 2\,\mathrm{Size}(\mathrm{Th}_2(\sqrt{n}))$$

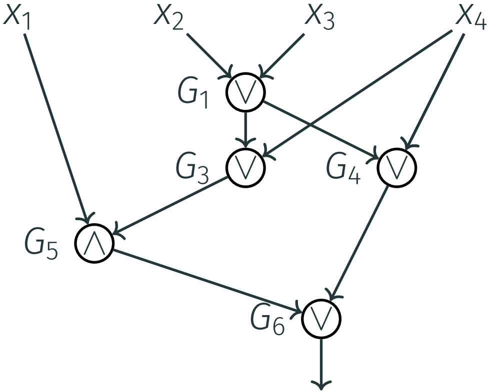
Th_2. Upper Bound



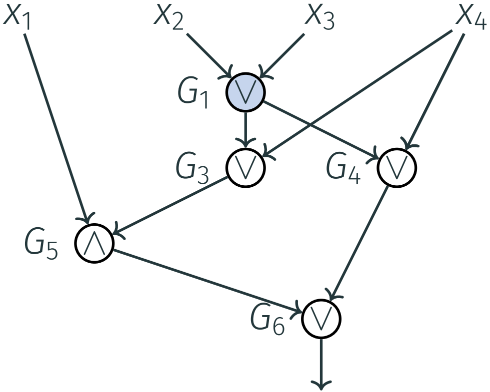
$$\mathrm{Th}_2(x_1, \ldots, x_n) = \mathrm{Th}_2(y_1, \ldots, y_{\sqrt{n}}) \ \text{Or} \ \mathrm{Th}_2(z_1, \ldots, z_{\sqrt{n}})$$

$$\mathrm{Size}(\mathrm{Th}_2(n)) \leq 2n + 2\,\mathrm{Size}(\mathrm{Th}_2(\sqrt{n})) \leq 2n + O(\sqrt{n})$$

Th_2. Lower Bound

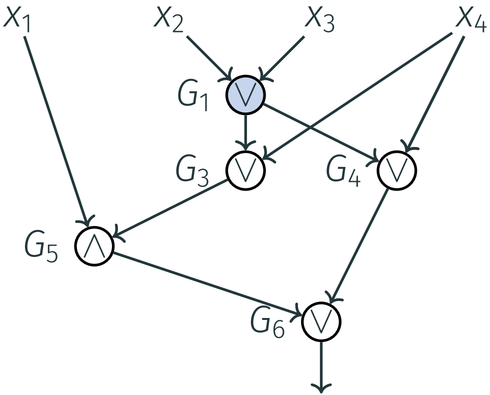


Th_2. Lower Bound



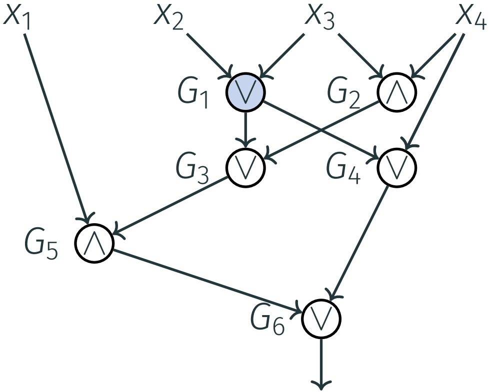
Th_2. Lower Bound

Case I:



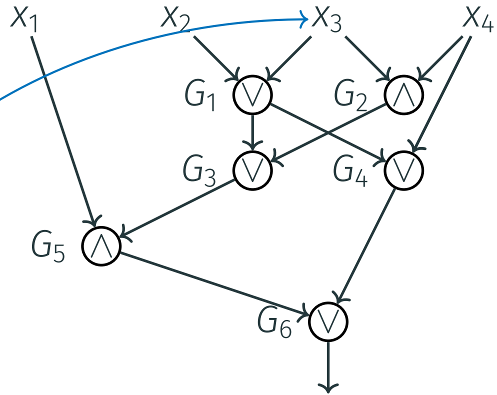
Th_2. Lower Bound

Case II:



Th$_2$. Lower Bound

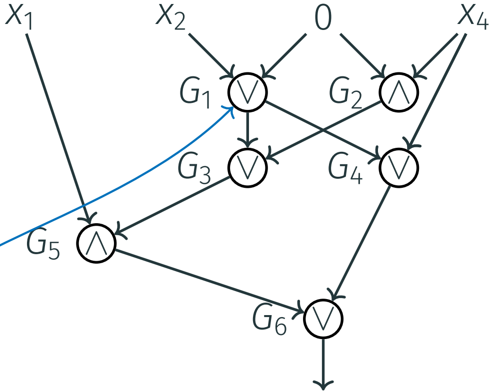
Case II:



assign $x_3 = 0$

Th_2. Lower Bound

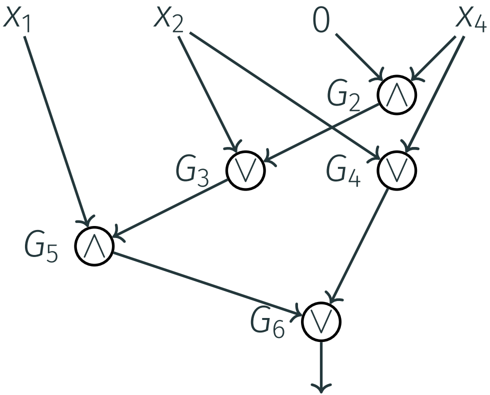
Case II:



G_1 now computes x_2

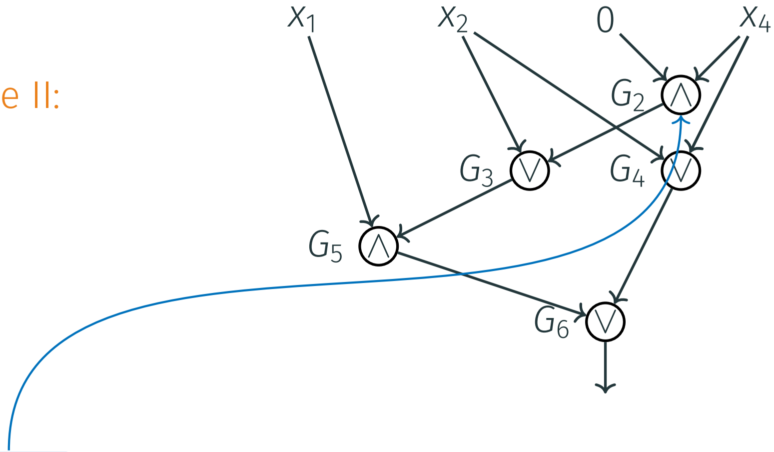
Th_2. Lower Bound

Case II:



Th_2. Lower Bound

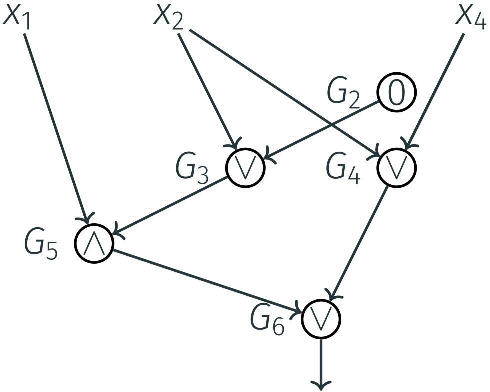
Case II:



$G_2 = 0$

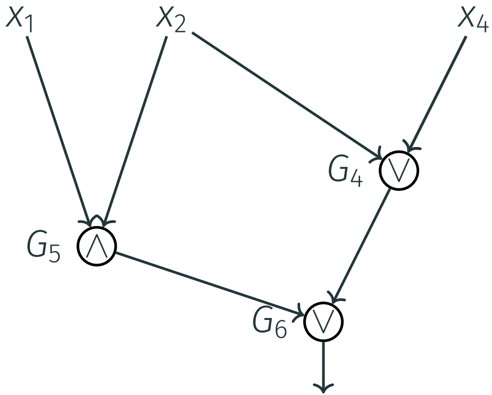
Th$_2$. Lower Bound

Case II:

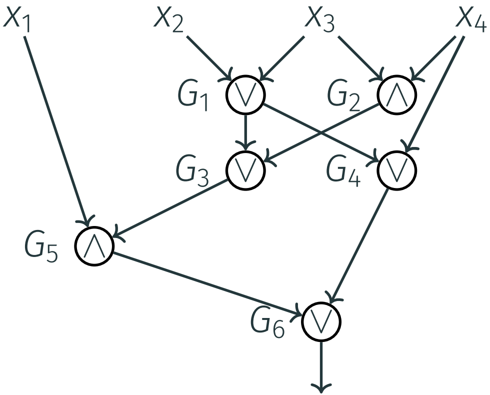


Th$_2$. Lower Bound

Case II:

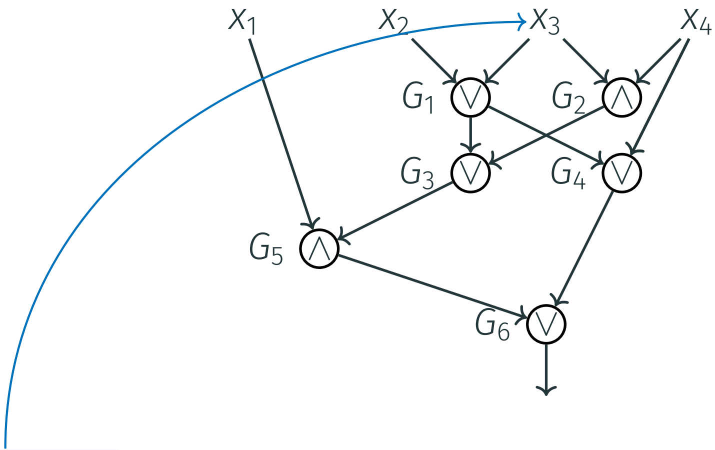


Th_2. Lower Bound



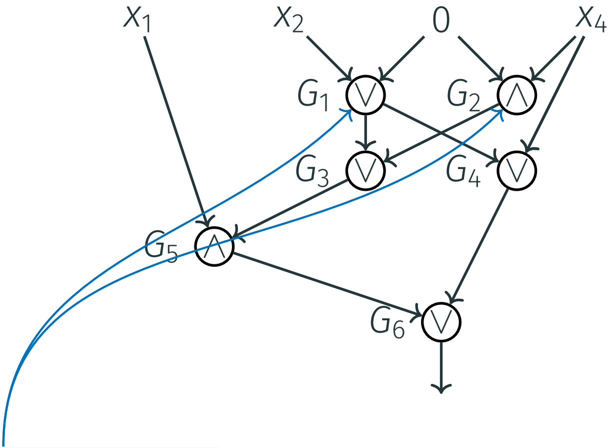
we start with circuit for $\mathbf{Th}_2^n$

Th_2. Lower Bound



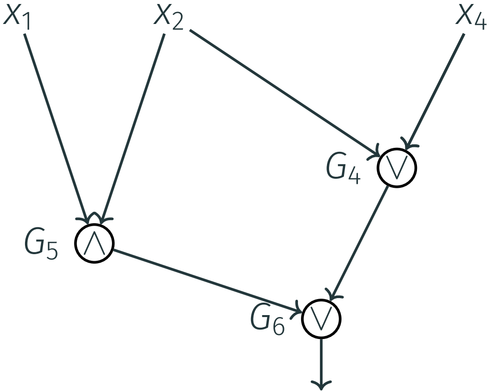
assign $x_3 = 0$

Th$_2$. Lower Bound



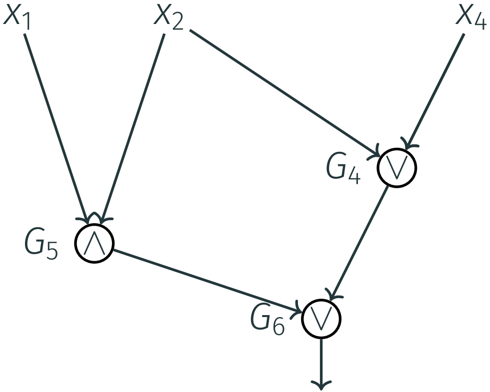
eliminate at least 2 gates

Th_2. Lower Bound



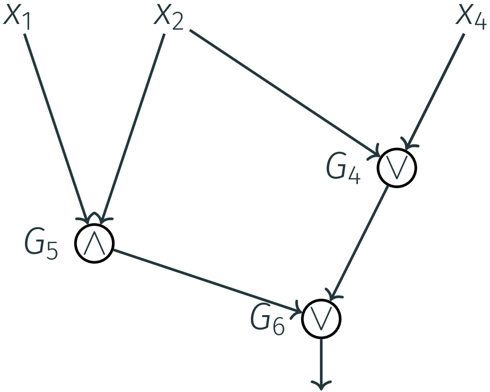
get a circuit for $\mathbf{Th}_2^{n-1}$

Th_2. Lower Bound



$$\mathrm{Size}(\mathrm{Th}_2^n) \geq 2 + \mathrm{Size}(\mathrm{Th}_2^{n-1})$$

Th_2. Lower Bound



$$\mathrm{Size}(\mathrm{Th}_2^n) \geq 2 + \mathrm{Size}(\mathrm{Th}_2^{n-1}) \geq 2n - O(1)$$