

A Note on Lower Bounds for Non-interactive Message Authentication Using Weak Keys

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Abstract

In this note, we prove lower bounds on the amount of entropy of random sources necessary for secure message authentication. We consider the problem of non-interactive c -time message authentication using a weak secret key having min-entropy k . We show that existing constructions using $(c + 1)$ -wise independent hash functions are optimal.

This result resolves one of the main questions left open by the work of Dodis and Spencer [2] who considered this problem for one-time message authentication of one-bit messages.

1 Introduction

1.1 Non-interactive Message Authentication

In this note, we revisit the problem of non-interactive message authentication: where Alice and Bob share a weak secret key $R \in \{0, 1\}^n$, and Alice wants to communicate up to c messages authentically to Bob over a channel controlled by the adversary Eve. This problem is known to have an easy solution with ε -security for $\varepsilon < 1$ using one of various possible universal hash functions, or more generally $c + 1$ -wise independent hash functions (see, for example, [6, 5] that give construction for $c = 1$). These solutions, however, require that the min-entropy $\mathbf{H}_\infty(R)$ of the source R is at least $\frac{cn}{c+1} + \log(\frac{1}{\varepsilon})$.

Dodis and Spencer [2] studied this problem with the goal of finding a lower bound on the min-entropy of R . They showed that for any integer $k \geq \frac{n}{2}$, and any one-round message authentication protocol for one-bit messages, there exists a k -flat source R such that the advantage of the adversary in forging the tag is at least $2^{n/2-k}$, or in other words, $\mathbf{H}_\infty(R) \geq \frac{n}{2} + \log(\frac{1}{\varepsilon})$. This showed that the construction using universal hash functions is optimal for one-bit messages. However, the bound for many time message authentication is still far from optimal and this was left as one of the main open questions in [2]. Specifically, the authors state that it is interesting to extend their quantitative results for private-key encryption and especially authentication to larger than one-bit message spaces. While this question has subsequently been almost resolved for the case of private-key encryption [1], it has remained open for the case of private-key authentication.

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1.2 Our contribution and Comparison with [2]

We answer this open question in the affirmative, i.e., that for any integer $k \geq \frac{cn}{c+1}$, and any c -round message authentication protocol, there exists a k -flat source R such that the advantage of the adversary in forging the tag is at least $2^{cn/(c+1)-k}$, or in other words, $\mathbf{H}_\infty(R) \geq \frac{cn}{c+1} + \log(\frac{1}{\epsilon})$. Our proof uses a simple idea based on the chain rule for Shannon entropy.

In comparison, the result of [2] was proved by considering a bipartite multigraph with the edges corresponding to the keys and the vertices on each part corresponding to the tags of the bit 0 and 1, respectively. They then partitioned their proof into two cases (i) where there are few tags corresponding to the bit 0, in which case it is easy to guess $\text{Tag}(0, R)$, and (ii) where there are many tags corresponding to the bit 0, but where knowing $\text{Tag}(0, R)$ gives significant information about $\text{Tag}(1, R)$. It seems that one might be able to generalize this idea to prove a lower bound for c -time message authentication by considering $c + 1$ cases as opposed to considering two cases for $c = 1$. However, the case analysis becomes significantly more involved due to the combinatorial nature of the proof, and perhaps this is a reason why the question has remained open for so long.

2 Preliminaries

For a set S , we let U_S denote the uniform distribution over S . For an integer $m \in \mathbb{N}$, we let U_m denote the uniform distribution over $\{0, 1\}^m$, the bit-strings of length m . For a distribution or random variable X we write $x \leftarrow X$ to denote the operation of sampling a random x according to X . For a set S , we write $s \leftarrow S$ as shorthand for $s \leftarrow U_S$.

2.1 Entropy Definitions

The *prediction probability* of a random variable X is defined as

$$\text{Pred}(X) := \max_x \Pr[X = x].$$

The *min-entropy* of X is defined as

$$\mathbf{H}_\infty(X) := -\log \text{Pred}(X).$$

We say that a random variable X is an (n, k) -source if $X \in \{0, 1\}^n$ and $\mathbf{H}_\infty(X) \geq k$. We also define *conditional prediction probability* of a random variable X conditioned on another random variable Z as

$$\begin{aligned} \text{Pred}(X|Z) &:= \mathbf{E}_{z \leftarrow Z} \left[\max_x \Pr[X = x|Z = z] \right] \\ &= \mathbf{E}_{z \leftarrow Z} \left[2^{-\mathbf{H}_\infty(X|Z=z)} \right]. \end{aligned}$$

The *conditional min-entropy* of X is defined as

$$\mathbf{H}_\infty(X|Z) := -\log \text{Pred}(X|Z).$$

Also, the Shannon entropy $\mathbf{H}_1(X)$ of a random variable X is defined as

$$\mathbf{H}_1(X) := -\sum_x \Pr[X = x] \log \Pr[X = x].$$

The conditional Shannon entropy of a random variable X conditioned on another random variable Z is defined as

$$\begin{aligned} \mathbf{H}_1(X|Z) &:= \mathbb{E}_{z \leftarrow Z} \mathbf{H}_1(X|Z = z) \\ &= -\mathbb{E}_{z \leftarrow Z} \sum_x \Pr[X = x|Z = z] \log \Pr[X = x|Z = z]. \end{aligned}$$

We will need the following standard facts about (conditional) min-entropy, and (conditional) Shannon entropy.

Fact 1. *Let X, Y, Z be arbitrary random variables, and let f be an arbitrary function. Then the following hold*

1. $\mathbf{H}_\infty(X|Z) \geq \mathbf{H}_\infty(f(X)|Z)$, and $\mathbf{H}_1(X|Z) \geq \mathbf{H}_1(f(X)|Z)$.
2. $\mathbf{H}_1(X, Y|Z) = \mathbf{H}_1(X|Y, Z) + \mathbf{H}_1(Y|Z)$.
3. $\mathbf{H}_1(X|Z) \geq \mathbf{H}_\infty(X|Z)$.

We remark here that the definition of the conditional Shannon entropy is fairly standard, but there are other alternative definitions in the literature for conditional min-entropy. However, our proposed definition is by now fairly standard. We direct the reader to [3] which contains a comprehensive discussion on conditional entropies, and proves Fact 1 among several other results.

2.2 Message Authentication Codes

In order to define a message authentication code, we first introduce the following game $\mathbf{G}_c(r)$. For a given function $\text{Tag} : \mathcal{M} \times \{0, 1\}^n \mapsto \mathcal{T}$ and a fixed secret key $r \in \{0, 1\}^n$, an adversary Eve is allowed to make at most c adaptive queries μ_1, \dots, μ_c to $\text{Tag}(\cdot, r)$. We say that Eve wins the game if she outputs a pair (μ_{c+1}, σ) , such that $\text{Tag}(\mu_{c+1}, r) = \sigma$ and $\mu_{c+1} \notin \{\mu_1, \dots, \mu_c\}$. We define the advantage of Eve in this game as

$$\text{Adv}_c^{\text{Eve}}(r) = \Pr[\text{Eve wins } \mathbf{G}_c(r)].$$

Definition 1. *A function $\text{Tag} : \mathcal{M} \times \{0, 1\}^n \mapsto \mathcal{T}$ is called a c -time (n, k, ε) -secure message authentication code, if for any distribution R on $\{0, 1\}^n$ with $\mathbf{H}_\infty(R) \geq k$, for any computationally unbounded adversary Eve,*

$$\mathbb{E}_{r \leftarrow R} [\text{Adv}_c^{\text{Eve}}(r)] \leq \varepsilon.$$

2.3 k -wise Independent Hash Functions

Here we define and give a well-known construction of k -wise independent hash functions.

Definition 2. *A function $H : \mathcal{X} \times \mathcal{R} \mapsto \mathcal{Y}$ is said to be a k -wise independent hash function if for all $y_1, \dots, y_k \in \mathcal{Y}$, and all distinct $x_1, \dots, x_k \in \mathcal{X}$,*

$$\Pr_{r \leftarrow \mathcal{R}} (H(x_1, r) = y_1 \wedge \dots \wedge H(x_k, r) = y_k) = \frac{1}{|\mathcal{Y}|^k}.$$

Lemma 1 (folklore). *Let k be a positive integer, and let $\mathcal{X} = \mathcal{Y} = \mathbb{F}$, and $\mathcal{R} = \mathbb{F}^k$ for some finite field \mathbb{F} . Then the function $H : \mathcal{X} \times \mathcal{R} \mapsto \mathcal{Y}$ given by*

$$H(x, (r_0, \dots, r_{k-1})) := r_0 + r_1 \cdot x + \dots + r_{k-1} \cdot x^{k-1}$$

is a k -wise independent hash function.

3 Tight Bound for c -time MACs

In this section, we prove a lower bound on the error-probability ε for c -time message authentication protocol for deterministic functions Tag .

Theorem 1. *Let Tag be a c -time (n, k, ε) -secure message authentication code where $\text{Tag} : \mathcal{M} \times \{0, 1\}^n \mapsto \mathcal{T}$. Then we have the following.*

1. If $k \leq \frac{cn}{c+1}$ then $\varepsilon = 1$;
2. If $k > \frac{cn}{c+1}$ then $\varepsilon \geq 2^{\frac{cn}{c+1}-k}$.

Proof. Let U be an n -bit uniformly random string, and let $\mu_1, \dots, \mu_{c+1} \in \mathcal{M}$ be fixed distinct messages. Note that $\mathbf{H}_1(U) = n$. Using Fact 1 multiple times, we get

$$\begin{aligned}
 n = \mathbf{H}_1(U) &\geq \mathbf{H}_1(\text{Tag}(\mu_1, U), \dots, \text{Tag}(\mu_{c+1}, U)) \\
 &= \mathbf{H}_1(\text{Tag}(\mu_1, U)) + \mathbf{H}_1(\text{Tag}(\mu_2, U), \dots, \text{Tag}(\mu_{c+1}, U) | \text{Tag}(\mu_1, U)) \\
 &= \dots \\
 &= \sum_{i=1}^{c+1} \mathbf{H}_1(\text{Tag}(\mu_i, U) | \text{Tag}(\mu_1, U), \dots, \text{Tag}(\mu_{i-1}, U)) \\
 &\geq \sum_{i=1}^{c+1} \mathbf{H}_\infty(\text{Tag}(\mu_i, U) | \text{Tag}(\mu_1, U), \dots, \text{Tag}(\mu_{i-1}, U)) .
 \end{aligned}$$

Therefore, there exists $i \in \{1, \dots, c+1\}$, such that

$$\mathbf{H}_\infty(\text{Tag}(\mu_i, U) | \text{Tag}(\mu_1, U), \dots, \text{Tag}(\mu_{i-1}, U)) \leq \frac{n}{c+1} .$$

We fix an i satisfying this inequality. For any $\mathbf{t} = (t_1, \dots, t_{i-1}) \in \mathcal{T}^{i-1}$, let $\mathcal{E}(\mathbf{t})$ be a shorthand for the event that $\text{Tag}(\mu_j, U) = t_j$ for $1 \leq j < i$. From the definition of conditional min-entropy, we get the following.

$$\begin{aligned}
 2^{-\frac{n}{c+1}} &\leq \mathbb{E}_{\mathbf{t} \in \mathcal{T}^{i-1}} \max_{t_i \in \mathcal{T}} \Pr[\text{Tag}(\mu_i, U) = t_i | \mathcal{E}(\mathbf{t})] \\
 &= \sum_{\mathbf{t} \in \mathcal{T}^{i-1}} \Pr[\mathcal{E}(\mathbf{t})] \cdot \max_{t_i \in \mathcal{T}} \Pr[\text{Tag}(\mu_i, U) = t_i | \mathcal{E}(\mathbf{t})] \\
 &= \sum_{\mathbf{t} \in \mathcal{T}^{i-1}} \max_{t_i \in \mathcal{T}} \Pr[\text{Tag}(\mu_j, U) = t_j \text{ for } 1 \leq j \leq i] . \tag{1}
 \end{aligned}$$

For every fixed $\mathbf{t} = (t_1, \dots, t_{i-1}) \in \mathcal{T}^{i-1}$, let $\mu_{\mathbf{t}}$ be the most probable value of $\text{Tag}(\mu_i, U)$ given $\text{Tag}(\mu_j, U) = t_j$ for $1 \leq j < i$. Intuitively, we want to choose a distribution over the set of keys so that $\text{Tag}(\mu_j, U) = t_j$ for $1 \leq j < i$ implies that $\text{Tag}(\mu_i, U) = \mu_{\mathbf{t}}$. Then, given tags for μ_1, \dots, μ_{i-1} , we can always guess the tag for μ_i . Let $\mathcal{K}_{\mathbf{t}}$ be the set of keys corresponding to $\mu_{\mathbf{t}}$, i.e.,

$$\mathcal{K}_{\mathbf{t}} = \{r \in \{0, 1\}^n | \text{Tag}(\mu_i, r) = \mu_{\mathbf{t}}, \text{Tag}(\mu_j, r) = t_j \text{ for } 1 \leq j < i\} .$$

Let also

$$\mathcal{K} = \bigcup_{\mathbf{t} \in \mathcal{T}^{i-1}} \mathcal{K}_{\mathbf{t}} .$$

From inequality (1),

$$|\mathcal{K}| \geq 2^n \cdot 2^{\frac{-n}{c+1}} = 2^{\frac{cn}{c+1}}.$$

If $2^k \leq |\mathcal{K}|$, then let \mathcal{R} be an arbitrary 2^k element subset of \mathcal{K} . Otherwise, let

$$\mathcal{R} = \mathcal{K} \cup \mathcal{K}',$$

where \mathcal{K}' is a set of arbitrary keys from the set $\{0, 1\}^n \setminus \mathcal{K}$, such that $|\mathcal{R}| = 2^k$.

We claim that if R is uniformly distributed on \mathcal{R} , then there exists a strategy for **Eve** such that the advantage in guessing $\mathbf{Tag}(\mu_i, r)$ given $\mathbf{Tag}(\mu_1, r), \dots, \mathbf{Tag}(\mu_{i-1}, r)$ is at least $2^{\frac{cn}{n+1}-k}$ if $k > \frac{cn}{n+1}$, and 1, otherwise. To see this, notice that for any $r \in \mathcal{K}$, there is a unique value of $\mathbf{Tag}(\mu_i, r)$ given $\mathbf{Tag}(\mu_1, r), \dots, \mathbf{Tag}(\mu_{i-1}, r)$. Let the strategy of **Eve** be to guess this unique tag assuming $R \in \mathcal{K}$. Then, **Eve** succeeds with probability 1 if $R \in \mathcal{K}$, and hence the advantage of **Eve** is

$$\varepsilon \geq \frac{|\mathcal{R} \cap \mathcal{K}|}{2^k} \geq \frac{\min\left(2^k, 2^{\frac{cn}{c+1}}\right)}{2^k}.$$

The statement of the theorem now follows. \square

It is well-known that the bound from Theorem 1 can be achieved by using a family of $(c+1)$ -wise independent hash functions (see [4] for similar results). For the sake of completeness, we present this construction below.

Lemma 2 (folklore). *Let \mathbb{F} be a finite field, and let $\mathcal{M} = \mathcal{T} = \mathbb{F}$, and let the set of keys be \mathbb{F}^{c+1} with $n = (c+1) \log |\mathbb{F}|$. Then the function $\mathbf{Tag} : \mathcal{M} \times \mathbb{F}^{c+1} \mapsto \mathcal{T}$ defined as:*

$$\mathbf{Tag}(\mu, (r_0, \dots, r_c)) := r_0 + r_1 \cdot \mu + \dots + r_c \cdot \mu^c$$

is a c -time $(n, k, 2^{\frac{cn}{c+1}-k})$ -secure message authentication code.

Proof. Let U be uniform in \mathbb{F}^{c+1} . For any fixed strategy of **Eve**, and $r \in \mathbb{F}^{c+1}$, let $f(r)$ denote $\mathbf{Adv}_c^{\mathbf{Eve}}(r)$. Let μ_1, \dots, μ_{c+1} be arbitrary distinct messages in \mathcal{M} . By Lemma 1, we have that for any $\sigma \in \mathcal{T}$, the probability that $\mathbf{Tag}(\mu_{c+1}, U) = \sigma$ given $\mathbf{Tag}(\mu_1, U), \dots, \mathbf{Tag}(\mu_c, U)$ is at most $\frac{1}{|\mathbb{F}|} = 2^{-n/(c+1)}$. Hence,

$$\mathbb{E}_{r \leftarrow U}[f(r)] \leq 2^{-\frac{n}{c+1}}.$$

Now, consider a random key $R \in \mathbb{F}^{c+1}$, such that $\mathbf{H}_\infty(R) \geq k$. Then

$$\begin{aligned} \mathbb{E}_{r \leftarrow R}[f(r)] &= \sum_{r \in \mathbb{F}^{c+1}} \Pr(R = r) \cdot f(r) \\ &\leq \max_{r \in \mathbb{F}^{c+1}} \Pr(R = r) \sum_{r \in \mathbb{F}^{c+1}} f(r) \\ &\leq 2^{-k} \cdot 2^n \cdot \mathbb{E}_{r \leftarrow U}[f(r)] \\ &\leq 2^{n-k} \cdot 2^{-\frac{n}{c+1}} \\ &= 2^{\frac{cn}{c+1}-k}, \end{aligned}$$

as needed. \square

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