

# Circuit size lower bounds and #SAT upper bounds through a general framework\*

Alexander Golovnev<sup>†</sup>    Alexander S. Kulikov<sup>‡</sup>    Alexander V. Smal<sup>§</sup>  
Suguru Tamaki<sup>¶</sup>

## Abstract

Most of the known *lower bounds* for binary Boolean circuits with unrestricted depth are proved by the gate elimination method. The most efficient known *algorithms* for the #SAT problem on binary Boolean circuits use similar case analyses to the ones in gate elimination. Chen and Kabanets recently showed that the known case analyses can also be used to prove *average case circuit lower bounds*, that is, lower bounds on the size of approximations of an explicit function.

In this paper, we provide a general framework for proving worst/average case lower bounds for circuits and upper bounds for #SAT that is built on ideas of Chen and Kabanets. A proof in such a framework goes as follows. One starts by fixing three parameters: a class of circuits, a circuit complexity measure, and a set of allowed substitutions. The main ingredient of a proof goes as follows: by going through a number of cases, one shows that for any circuit from the given class, one can find an allowed substitution such that the given measure of the circuit reduces by a sufficient amount. This case analysis immediately implies an upper bound for #SAT. To obtain worst/average case circuit complexity lower bounds one needs to present an explicit construction of a function that is a disperser/extractor for the class of sources defined by the set of substitutions under consideration.

We show that many known proofs (of circuit size lower bounds and upper bounds for #SAT) fall into this framework. Using this framework, we prove the following new bounds: average case lower bounds of  $3.24n$  and  $2.59n$  for circuits over  $U_2$  and  $B_2$ , respectively (though the lower bound for the basis  $B_2$  is given for a quadratic disperser whose explicit construction is not currently known), and faster than  $2^n$  #SAT-algorithms for circuits over  $U_2$  and  $B_2$  of size at most  $3.24n$  and  $2.99n$ , respectively. Here by  $B_2$  we mean the set of all bivariate Boolean functions, and by  $U_2$  the set of all bivariate Boolean functions except for parity and its complement.

---

\*The research presented in Section 3.1 is supported in part by MEXT KAKENHI (24106003); JSPS KAKENHI (26330011, 16H02782); the John Mung Advanced Program of Kyoto University. The research presented in Section 3.2 is partially supported by NSF grant 1319051. The research presented in Sections 4–5 is supported by Russian Science Foundation (project 16-11-10123). Part of the work was performed while Suguru Tamaki was at Department of Computer Science and Engineering, University of California, San Diego and the Simons Institute for the Theory of Computing, Berkeley.

<sup>†</sup>New York University, St. Petersburg Department of Steklov Institute of Mathematics of the Russian Academy of Sciences [alexgolovnev@gmail.com](mailto:alexgolovnev@gmail.com).

<sup>‡</sup>St. Petersburg Department of Steklov Institute of Mathematics of the Russian Academy of Sciences [kulikov@logic.pdmi.ras.ru](mailto:kulikov@logic.pdmi.ras.ru).

<sup>§</sup>St. Petersburg Department of Steklov Institute of Mathematics of the Russian Academy of Sciences [smal@logic.pdmi.ras.ru](mailto:smal@logic.pdmi.ras.ru).

<sup>¶</sup>Kyoto University [tamak@kuis.kyoto-u.ac.jp](mailto:tamak@kuis.kyoto-u.ac.jp).

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	Background . . . . .	1
1.2	Techniques and Results . . . . .	2
1.3	Framework . . . . .	3
<b>2</b>	<b>Preliminaries</b>	<b>4</b>
2.1	Boolean Functions . . . . .	4
2.2	Dispersers and Extractors . . . . .	4
2.3	Circuits . . . . .	5
2.4	Circuit Normalization . . . . .	5
2.5	Circuit Complexity Measures . . . . .	6
2.6	Splitting Numbers and Splitting Vectors . . . . .	7
2.7	Azuma’s Inequality . . . . .	9
<b>3</b>	<b>Toolkit</b>	<b>10</b>
3.1	Main Theorem . . . . .	10
3.2	Discussion . . . . .	12
<b>4</b>	<b>Bounds for the Basis <math>U_2</math></b>	<b>13</b>
4.1	Bit Fixing Substitutions . . . . .	13
4.2	Projections . . . . .	14
<b>5</b>	<b>Bounds for the Basis <math>B_2</math></b>	<b>17</b>
5.1	Affine Substitutions . . . . .	17
5.2	Quadratic Substitutions . . . . .	19
<b>6</b>	<b>Open Problems</b>	<b>23</b>

## 1 Introduction

### 1.1 Background

In this paper, we study binary Boolean circuits with no restriction on the depth. This is a natural model for computing Boolean functions that can be viewed as a simple program where each instruction is just a binary Boolean operation. Shannon [57] showed that for almost all Boolean functions of  $n$  variables the size of a smallest circuit (equivalently, the minimal number of instructions) computing this function is  $\Omega(2^n/n)$ . The proof is based on a counting argument (the number  $2^{2^n}$  of all functions of  $n$  variables is larger than the number of circuits of size  $o(2^n/n)$ ) and, for this reason, does not give an explicit function of high circuit complexity. By saying “explicit” one usually means a function from NP. Showing a superpolynomial lower bound for an explicit function would imply  $P \neq NP$ . However, despite of many efforts [54, 48, 58, 10, 22, 29, 68, 5], currently we have only small linear lower bounds:  $(3 + 1/86)n - o(n)$  for the full binary basis  $B_2$  consisting of all binary Boolean functions [27] and  $5n - o(n)$  for the basis  $U_2$  consisting of all binary Boolean functions except for parity and its complement [39, 33].

Going to larger complexity classes, it is known that the classes MA/1 [51],  $S_2^p$  [12], and  $P^{\text{prMA}}$  [13] require circuits of superlinear size and the class MAEXP [11] has superpolynomial circuit complexity. Proving a superlinear lower bound on the circuit complexity of  $E^{\text{NP}}$  remains to be a major open problem.

Recently, Williams [62, 65] presented the following approach to prove circuit size lower bounds against  $E^{\text{NP}}$  or NE using SAT-algorithms: a super-polynomially faster than  $2^n$  algorithm for the circuit satisfiability problem of a “reasonable” circuit class  $\mathcal{C}$  implies either  $E^{\text{NP}} \not\subseteq \mathcal{C}$  or  $NE \not\subseteq \mathcal{C}$ , depending on  $\mathcal{C}$  and the running time of the algorithm. The approach has been strengthened and simplified by subsequent work [61, 66, 64, 9, 34], see also excellent surveys [53, 46, 63] on this topic.

Williams’ result inspired lots of work on satisfiability algorithms for various circuit classes [32, 64, 17, 4, 3, 44, 18, 59, 60, 1, 42]. In addition to satisfiability algorithms, several papers [52, 31, 6, 55, 16, 14, 19, 50] also obtained average-case lower bounds (also known as correlation bounds, see [36, 37, 30]) by investigating the analysis of algorithms instead of just applying Williams’ result for worst-case lower bounds. In particular, Chen and Kabanets [15] presented algorithms that count the number of satisfying assignments of circuits over  $U_2$  and  $B_2$  and run in time exponentially faster than  $2^n$  if input instances have at most  $2.99n$  and  $2.49n$  gates, respectively (improving also the previously best known #SAT-algorithm by Nurk [45]). At the same time, they showed that  $2.99n$  sized circuits over  $U_2$  and  $2.49n$  sized circuits over  $B_2$  have exponentially small correlations with the parity function and affine extractors having “good” parameters, respectively.

To prove a lower bound of  $\zeta n$  on the circuit size, one usually shows that for any circuit there is a substitution  $x_i \leftarrow f$  eliminating at least  $\zeta$  gates from the circuit. For example, to prove a lower bound of  $3n - 3$  on the circuit size over  $U_2$  of the parity function, Schnorr [54] shows how to make a bit-fixing substitution (i.e.,  $f = c$  for  $c \in \{0, 1\}$ ) eliminating at least 3 gates from any  $U_2$ -circuit. Demenkov and Kulikov [22] prove a lower bound of  $3n - o(n)$  on the circuit size over  $B_2$  of an affine disperser by showing that for any  $B_2$ -circuit there is an affine substitution ( $f = \bigoplus_{j \in J} x_j \oplus c$ ) eliminating at least three gates from the circuit. Chen and Kabanets proved new average case lower bounds and #SAT upper bounds by analyzing what happens in the complementary branch  $x_i \leftarrow f \oplus 1$  of proofs by Schnorr and by Demenkov and Kulikov.

## 1.2 Techniques and Results

The main qualitative contribution of this paper is a general framework for proving circuit worst/average case lower bounds and #SAT upper bounds. This framework is separated into conceptual and technical parts. The conceptual part is a proof that for a given circuit complexity measure and a set of allowed substitutions, for any circuit, there is a substitution that reduces the complexity of the circuit by a sufficient amount. This is usually shown by analyzing the structure of the top of a circuit. The technical part is a set of lemmas that allows us to derive worst/average case circuit size lower bounds and #SAT upper bounds as one-line corollaries from the corresponding conceptual part. The technical part can be used in a black-box way: given a proof that reduces the complexity measure of a circuit (conceptual part), the technical part implies circuit lower bounds and #SAT upper bounds. For example, by plugging in the proofs by Schnorr and by Demenkov and Kulikov, one immediately gets the bounds given by Chen and Kabanets. We also give new proofs that lead to the quantitatively better results.

The main quantitative contribution of the paper is the following new bounds which are currently the strongest known bounds:

- average case lower bounds of  $3.24n$  and  $2.59n$  for circuits over  $U_2$  and  $B_2$  (though the lower bound for the basis  $B_2$  is given for a quadratic disperser whose explicit construction is not currently known), respectively, improving upon the bounds of  $2.99n$  and  $2.49n$  [15];
- faster than  $2^n$  #SAT-algorithms for circuits over  $U_2$  and  $B_2$  of size at most  $3.24n$  and  $2.99n$ , respectively, improving upon the bounds of  $2.99n$  and  $2.49n$  [15].

### 1.3 Framework

We prove circuit lower bounds (both in the worst case and in the average case) and upper bounds for #SAT using the following four step framework.

**Initial setting** We start by specifying the three main parameters: a class of circuits  $\mathcal{C}$ , a set  $\mathcal{S}$  of allowed substitutions, and a circuit complexity measure  $\mu$ . A set of allowed substitutions naturally defines a class of “sources”. For the circuit lower bounds we consider functions that are non-constant (dispersers) or close to uniform (extractors) on corresponding sets of sources. In this paper we focus on the following four sets of substitutions where each set extends the previous one:

1. Bit fixing substitutions,  $\{x_i \leftarrow c\}$ : substitute variables by constants.
2. Projections,  $\{x_i \leftarrow c, x_i \leftarrow x_j \oplus c\}$ : substitute variables by constants and other variables and their negations.
3. Affine substitutions,  $\{x_i \leftarrow \bigoplus_{j \in J} x_j \oplus c\}$ : substitute variables by affine functions of other variables.
4. Quadratic substitutions,  $\{x_i \leftarrow p : \deg(p) \leq 2\}$ : substitute variables by degree two polynomials of other variables.

**Case analysis** We then prove the main technical result stating that for any circuit from the class  $\mathcal{C}$  there exists (and can be constructed efficiently) an allowed substitution  $x_i \leftarrow f \in \mathcal{S}$  such that the measure  $\mu$  is reduced by a sufficient amount under both substitutions  $x_i \leftarrow f$  and  $x_i \leftarrow f \oplus 1$ .

**#SAT upper bounds** As an immediate consequence, we obtain an upper bound on the running time of an algorithm solving #SAT for circuits from  $\mathcal{C}$ . The corresponding algorithm takes as input a circuit, branches into two cases  $x_i \leftarrow f$  and  $x_i \leftarrow f \oplus 1$ , and proceeds recursively. When applying a substitution  $x_i \leftarrow f \oplus c$ , it replaces all occurrences of  $x_i$  by a subcircuit computing  $f \oplus c$ . The case analysis provides an upper bound on the size of the resulting recursion tree.

**Circuit size lower bounds** Then, by taking a function that survives under sufficiently many allowed substitutions, we obtain lower bounds on the average case and worst case circuit complexity of the function. Below, we describe such functions, i.e., dispersers and extractors for the classes of sources under consideration.

1. The class of bit fixing substitutions generates the class of *bit-fixing sources* [20]. Extractors for bit-fixing sources find many applications in cryptography (see [24] for an excellent survey of the topic). The standard function that is a good disperser and extractor for such sources is the parity function  $x_1 \oplus \dots \oplus x_n$ .

2. Projections define the class of *projection sources* [47]. Dispersers for projections are used to prove lower bounds for depth-three circuits [47]. It is shown [47] that a binary BCH code with appropriate parameters is a disperser for  $n - o(n)$  substitutions. See [49] for an example of extractor with good parameters for projection sources.
3. Affine substitutions give rise to the class of *affine sources*. Dispersers and extractors for affine sources find applications in circuit lower bounds [15, 21, 22, 27, 55]. There are several known constructions of dispersers [8, 56] and extractors [67, 40, 7, 41] that are resistant to  $n - o(n)$  substitutions.
4. The class of quadratic substitutions generates a special case of *polynomial sources* [26, 7] and *quadratic varieties sources* [25]. An explicit construction of disperser for quadratic varieties sources would imply new circuit lower bounds [29]. Although an explicit construction of a function resistant to sufficiently many quadratic substitutions is not currently known, it is easy to show that a random function is resistant to any  $n - o(n)$  quadratic substitutions.

## 2 Preliminaries

### 2.1 Boolean Functions

We denote by  $B_n$  the set of all  $n$ -variate Boolean functions and define  $U_2 = B_2 \setminus \{\oplus, \equiv\}$  as the set of all *binary* Boolean functions except for parity and its complement.

The set of all sixteen binary Boolean functions  $f(x, y) \in B_2$  can be classified as follows: 1) two constant functions: 0 and 1; we also call them *trivial*; 2) four functions that depend essentially on one of the arguments only:  $x, x \oplus 1, y, y \oplus 1$ ; we call them *degenerate*; 3) eight *and-type* functions:  $(x \oplus a) \cdot (y \oplus b) \oplus c$  where  $a, b, c \in \{0, 1\}$ ; 4) two *xor-type* functions:  $x \oplus y \oplus a$ , where  $a \in \{0, 1\}$ .

Hence  $U_2$  consists of all binary functions except for xor-type functions. An important property of binary and-type functions  $(x \oplus a) \cdot (y \oplus b) \oplus c$ , useful for case analyses, is the following: one can turn this function into a constant  $c$  by assigning  $x \leftarrow a$  or  $y \leftarrow b$ .

### 2.2 Dispersers and Extractors

Let  $x_1, \dots, x_n$  be Boolean variables, and  $f \in B_{n-1}$  be a function of  $n - 1$  variables. We say that  $x_i \leftarrow f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$  is a substitution to the variable  $x_i$ .

Let  $g \in B_n$  be a function, then the *restriction* of  $g$  under the substitution  $f$  is a function  $h = (g|x_i \leftarrow f)$  of  $n - 1$  variables, such that

$$h(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = g(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n),$$

where  $x_i = f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ . Similarly, if  $K \subseteq \{0, 1\}^n$  is a subset of the Boolean cube, then the *restriction* of  $K$  under this substitution is  $K' = (K|x_i \leftarrow f)$ , such that  $(x_1, \dots, x_n) \in K'$  if and only if  $(x_1, \dots, x_n) \in K$  and  $x_i = f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ .

For a family of functions  $\mathcal{F} = \{f : \{0, 1\}^* \rightarrow \{0, 1\}\}$  we define a set of corresponding substitutions  $\mathcal{S}(\mathcal{F})$  that contains the following substitutions: for every  $1 \leq i \leq n, c \in \{0, 1\}, f \in \mathcal{F}$ ,  $\mathcal{S}$  contains the substitution  $x_i \leftarrow f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \oplus c$ .

Let  $\mathcal{S}$  be a set of substitutions. We say that a set  $K \subseteq \{0, 1\}^n$  is an  $(\mathcal{S}, n, r)$ -source<sup>1</sup> if it can be obtained from  $\{0, 1\}^n$  by applying at most  $r$  substitutions from  $\mathcal{S}$ .

A function  $f \in B_n$  is called an  $(\mathcal{S}, n, r)$ -disperser<sup>2</sup> if it is not constant on every  $(\mathcal{S}, n, r)$ -source. A function  $f \in B_n$  is called an  $(\mathcal{S}, n, r, \varepsilon)$ -extractor if  $|\Pr_{x \leftarrow K}[f(x) = 1] - 1/2| \leq \varepsilon$  for every  $(\mathcal{S}, n, r)$ -source  $K$ . Dispersers and extractors for  $\mathcal{S}$  being a set of affine (quadratic) substitutions are called *affine* (*quadratic*) and we omit  $\mathcal{S}$  in these cases.

### 2.3 Circuits

A circuit over the basis  $\Omega \subseteq B_2$  is a directed acyclic graph with the following properties: 1) the indegree of each node is either zero or two; 2) each node of zero indegree is labeled by a variable and is called an *input* or an *input gate*; 3) each node of indegree two is labeled with a binary Boolean function from  $\Omega$  called an *operation* of this gate; the node itself is called an *internal gate* or just a *gate*; 4) there is a unique node of outdegree zero and it is called an *output*. Such a circuit computes in a natural way a function from  $B_n$ , where  $n$  is the number of input gates of the circuit. In this paper, we consider circuits over the bases  $\Omega = B_2$  and  $\Omega = U_2$ .

An *xor-gate* (*and-gate*) is a gate computing an xor-type (and-type, respectively) operation. A *k-gate* (*k<sup>+</sup>-gate*) is a gate of outdegree exactly  $k$  (at least  $k$ , respectively).

For a circuit  $C$ , by  $s(C)$  we denote the *size* of  $C$ , that is, the number of internal gates of  $C$ . By  $i(C)$  and  $i_1(C)$  we denote the total number of input gates of  $C$  and the number of input 1-gates, respectively. For a function  $f \in B_n$ , by  $C_\Omega(f)$  we denote the minimal size of a circuit over  $\Omega$  computing  $f$ .

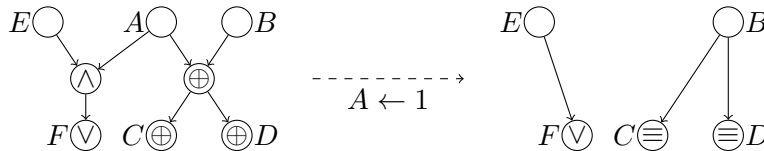
For two Boolean functions  $f, g \in B_n$ , the correlation between them is defined as

$$\text{Cor}(f, g) = \left| \Pr_{x \leftarrow \{0,1\}^n} [f(x) = g(x)] - \Pr_{x \leftarrow \{0,1\}^n} [f(x) \neq g(x)] \right| = 2 \left| \frac{1}{2} - \Pr_{x \leftarrow \{0,1\}^n} [f(x) \neq g(x)] \right|.$$

For a function  $f \in B_n$ , and  $0 \leq \varepsilon \leq 1$ , by  $C_\Omega(f, \varepsilon)$  we denote the minimal size of a circuit over  $\Omega$  computing function  $g$  such that  $\text{Cor}(f, g) \geq \varepsilon$ .

### 2.4 Circuit Normalization

When some gate  $A$  of a circuit trivializes (e.g., when an input gate is assigned a constant), some other gates (in particular, all successors of  $A$ ) become trivial or degenerate. Such gates can be eliminated from the circuit without changing the function computed by the circuit. We do not describe this process formally. Instead, we give an illustrating example.



This example, in particular, reveals the following properties:

<sup>1</sup>Usually in the literature a source corresponds to a distribution over a subset of  $\{0, 1\}^n$ . In this paper, we focus only on uniform distributions, so we associate a source with its support.

<sup>2</sup>In this paper, we consider only dispersers and extractors with one bit outputs.

1. when a gate is trivialized, all its successors are eliminated;
2. during this process, the out-degree of a gate may change;
3. the binary operation computed by a gate may also change (but an and-type gate is never turned into an xor-type gate, and vice versa).

A gate is called *useless* if it is a 1-gate and is fed by a predecessor of its successor:



In this case  $E$  actually computes a binary operation of  $A$  and  $B$  and this operation can be computed in the gate  $E$  directly. This might require to change an operation at  $E$  (if this circuit is over  $U_2$  then  $E$  still computes an and-type operation of  $A$  and  $B$  as an xor-type binary function requires three gates in  $U_2$ ).

By *normalizing* a circuit we mean removing all gates that compute trivial or degenerate operations and removing all useless gates.

In the proofs of the paper we implicitly assume that if two gates are fed by the same variable then either there is no wire between them or each of the gates feed also some other gate (otherwise, one of the gates would be useless) and hence we do not care about this wire between the gates.

## 2.5 Circuit Complexity Measures

A function  $\mu$  mapping circuits to non-negative real values is called a *circuit complexity measure* if for any circuit  $C$ ,

- normalization of  $C$  does not increase its measure, and
- if  $\mu(C) = 0$  then  $C$  has no gates.

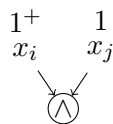
For a fixed circuit complexity measure  $\mu$ , and function  $f \in B_n$ , we define  $\mu(f)$  to be the minimum value of  $\mu(C)$  over circuits  $C$  computing  $f$ . Similarly, we define  $\mu(f, \varepsilon)$  to be the minimum value of  $\mu(C)$  over circuits  $C$  computing  $g$  such that  $\text{Cor}(f, g) \geq \varepsilon$ .

In this paper, we focus on the following two circuit complexity measures:

- $\mu(C) = s(C) + \alpha \cdot i(C)$  where  $\alpha \geq 0$  is a constant;
- $\mu(C) = s(C) + \alpha \cdot i(C) - \sigma \cdot i_1(C)$  where  $0 \leq \sigma \leq 1/2 < \alpha$  are constants.

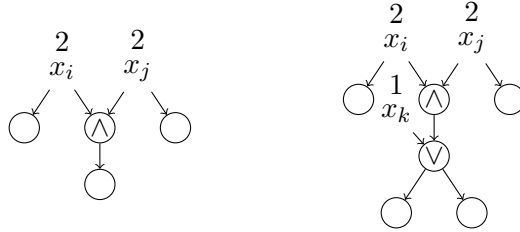
It is not difficult to see that these two functions are indeed circuit complexity measures. The condition  $0 \leq \sigma \leq 1/2 < \alpha$  is needed to guarantee that if by removing a degenerate gate we increase the out-degree of a variable, the measure does not increase (an example is given on the next page) and that the measure is always non-negative.

Intuitively we include the term  $i(C)$  into the measure to handle cases like the one below (throughout the paper, we use labels above the gates to indicate their outdegree):



In this case, by assigning  $x_i \leftarrow 0$  we make the circuit independent of  $x_j$ , so the measure is reduced by at least  $2\alpha$ . Usually, our goal is to show that we can find a substitution to a variable that eliminates at least some constant number  $k$  of gates, that is, to show a complexity decrease of at least  $k + \alpha$ . Thus, by choosing a large enough value of  $\alpha$  we can always guarantee that  $2\alpha \geq \alpha + k$ . Thus, in the case above we do not even need to count the number of gates eliminated under the substitution.

The measure  $\mu(C) = s(C) + \alpha \cdot i(C) - \sigma \cdot i_1(C)$  allows us to get an advantage of new 1-variables that are introduced during splitting.



For example, by assigning  $x_i \leftarrow 0$  in a situation like the one in the left picture we reduce the measure by at least  $3 + \alpha + \sigma$ . As usual, the advantage comes with a related disadvantage. If, for example, a closer look at the circuit from the left part reveals that it actually looks like as shown on the right, then by assigning  $x_i \leftarrow 0$  we introduce a new 1-variable  $x_j$ , but also lose one 1-variable (namely,  $x_k$  is now a 2-variable). Hence, in this case  $\mu$  is reduced only by  $(3 + \alpha)$  rather than  $(3 + \alpha + \sigma)$ . That is, our initial estimate was too optimistic. For this reason, when use a measure with  $i_1(C)$  we check carefully for each eliminated gate if this elimination increases the degree of a 1-variable.

## 2.6 Splitting Numbers and Splitting Vectors

Let  $\mu$  be a circuit complexity measure and  $C$  be a circuit. Consider a recursive algorithm solving #SAT on  $C$  by repeated substitutions. Assume that at the current step the algorithm chooses  $k$  variables  $x_1, \dots, x_k$  and  $k$  functions  $f_1, \dots, f_k$  to substitute these variables and branches into  $2^k$  possible situations:  $x_1 \leftarrow f_1 \oplus c_1, \dots, x_k \leftarrow f_k \oplus c_k$  for all possible  $c_1, \dots, c_k \in \{0, 1\}$  (in other words, it partitions the Boolean hypercube  $\{0, 1\}^n$  into  $2^k$  subsets).<sup>3</sup> For each substitution, we normalize the resulting circuit. Let us call the  $2^k$  circuits  $C_1, \dots, C_{2^k}$ . We say that the current step has a *splitting vector*  $v = (a_1, \dots, a_{2^k})$  w.r.t.  $\mu$  if for all  $i \in [2^k]$ ,  $\mu(C) - \mu(C_i) \geq a_i > 0$ . That is, the splitting vector gives a lower bound on the complexity decrease under the considered substitution. The *splitting number*  $\tau(v)$  is the unique positive root of the equation  $\sum_{i \in [2^k]} x^{-a_i} = 1$ .

Splitting vectors and numbers are heavily used to estimate the running time of recursive algorithms. Below we assume that  $k$  is bounded by a constant. In all the proofs of this paper either  $k = 1$  or  $k = 2$ , that is, we always estimate the effect of assigning either one or two variables. If an algorithm always splits with a splitting number at most  $\beta$  then its running time is bounded by  $O^*(\beta^{\mu(C)})$ .<sup>4</sup> To show this, one notes that the recursion tree of this algorithm is  $2^k$ -ary and

<sup>3</sup>Sometimes it is easier to consider vectors of length that is not a power of 2 too. For example, we can have a branching into three cases: one with one substituted variable, and two with two substituted variables. All the results from this paper can be naturally generalized to this case. For simplicity, we state the results for splitting vectors of length  $2^k$  only.

<sup>4</sup> $O^*$  suppresses factors polynomial in the input length.



$k = O(1)$ , so it suffices to estimate the number of leaves. The number of leaves  $T(\mu)$  satisfies the recurrence  $T(\mu) \leq \sum_{i \in [2^k]} T(\mu - a_i)$  which implies that  $T(\mu) = O(\tau(v)^\mu)$  (we assume also that  $T(\mu) = O(1)$  when  $\mu = O(1)$ ). See, e.g., [38] for a formal proof.

For a splitting vector  $v = (a_1, \dots, a_{2^k})$  we define the following related quantities:

$$\bar{v}_{\max} = \max_{i \in [2^k]} \left\{ \frac{a_i}{k} \right\}, \quad \bar{v}_{\min} = \min_{i \in [2^k]} \left\{ \frac{a_i}{k} \right\}, \quad \bar{v}_{\text{avg}} = \frac{\sum_{i \in [2^k]} a_i}{k 2^k}.$$

Intuitively,  $\bar{v}_{\max}$  ( $\bar{v}_{\min}$ ,  $\bar{v}_{\text{avg}}$ ) is a (lower bound for) the maximum (minimum, average, respectively) complexity decrease per single substitution.

We will need the following estimates for the splitting numbers. It is known that a balanced binary splitting vector is better than an unbalanced one:  $2^{1/a} = \tau(a, a) < \tau(a + b, a - b)$  for  $0 < b < a$  (see, e.g., [38]). There is a known upper bound on  $\tau(a, b)$ .

**Lemma 1.**  $\tau(a, b) \leq 2^{1/\sqrt{ab}}$ .

In the following lemma we provide an asymptotic estimate of their difference.

**Lemma 2.** Let  $a_1 > a_2 > 0$ ,  $a' = (a_1 + a_2)/2$  and  $\delta(b) = \tau(a_1 + b, a_2 + b) - 2^{\frac{1}{a'+b}}$ . Then,  $\delta(b) = O((a_1 - a_2)^2/b^3)$  as  $b \rightarrow \infty$ .

*Proof.* Let  $x = \tau(a_1 + b, a_2 + b)$ , then by definition we have

$$1 = \frac{1}{x^{a_1+b}} + \frac{1}{x^{a_2+b}} = \frac{x^{-(a_1-a_2)/2} + x^{(a_1-a_2)/2}}{x^{a'+b}}.$$

Since

$$x = 2^{\frac{1}{a'+b}} + \delta(b) = 1 + \frac{\ln 2}{a'+b} + \delta(b) + O\left(\frac{1}{(a'+b)^2}\right)$$

and

$$(1 + \epsilon)^{(a_1-a_2)/2} = 1 + (a_1 - a_2)\epsilon/2 + (a_1 - a_2)(a_1 - a_2 - 2)\epsilon^2/8 + O(\epsilon^3),$$

we have

$$x^{-(a_1-a_2)/2} + x^{(a_1-a_2)/2} = 2 + \frac{(a_1-a_2)^2}{4} \left( \frac{\ln 2}{a'+b} + \delta(b) \right)^2 + O\left( \left( \frac{\ln 2}{a'+b} + \delta(b) \right)^3 \right).$$

We also have

$$x^{a'+b} = 2 \left( 1 + \delta(b)/2^{\frac{1}{a'+b}} \right)^{a'+b} = 2 \left( 1 + (a' + b)\delta(b)/2^{\frac{1}{a'+b}} + O(\delta(b)^2) \right).$$

By the definition of  $x$ , we have

$$\lim_{b \rightarrow \infty} \frac{(a_1 - a_2)^2 \ln^2 2}{2b^2} / (2b\delta(b)) = 1.$$

This implies

$$\delta(b) = \frac{(a_1 - a_2)^2 \ln^2 2}{4b^3} + o(1/b^3).$$

□

## 2.7 Azuma's Inequality

Following the approach from [15], we use a variant of Azuma's inequality with one-sided boundedness condition in order to obtain average case lower bounds. The standard version of Azuma's inequality requires the difference between two consecutive variables to be bounded, [15] considers the case when the difference takes on only two values but is bounded only from one side. For our results, we need a slightly more general variant of the inequality: the difference between two consecutive variables takes on up to  $k$  values and is bounded from one side. We provide a proof of this inequality, which is just an adjustment of proofs from [43, 2, 15].

A sequence  $X_0, \dots, X_m$  of random variables is a *supermartingale* if for every  $0 \leq i < m$ ,  $\mathbb{E}[X_{i+1}|X_i, \dots, X_0] \leq X_i$ .

**Lemma 3.** *Let  $X_0, \dots, X_m$  be a supermartingale, let  $Y_i = X_i - X_{i-1}$ . If  $Y_i \leq c$  and for fixed values of  $(X_0, \dots, X_{i-1})$ , the random variable  $Y_i$  is distributed uniformly over at most  $k \geq 2$  (not necessarily distinct) values, then for every  $\lambda \geq 0$ :*

$$\Pr[X_m - X_0 \geq \lambda] \leq \exp\left(\frac{-\lambda^2}{2mc^2(k-1)^2}\right).$$

*Proof.* For any  $t > 0$ ,

$$\begin{aligned} \Pr[X_m - X_0 \geq \lambda] &= \Pr\left[\sum_{i=1}^m Y_i \geq \lambda\right] \\ &= \Pr\left[\exp\left(t \cdot \sum_{i=1}^m Y_i\right) \geq e^{\lambda t}\right] \leq e^{-\lambda t} \cdot \mathbb{E}\left[\exp\left(t \cdot \sum_{i=1}^m Y_i\right)\right]. \end{aligned}$$

First we show that for any  $t > 0$ ,  $\mathbb{E}[e^{tY_i}] \leq \exp(t^2 c^2 (k-1)^2 / 2)$ . Since  $\{X_i\}$  is a supermartingale,  $\mathbb{E}[Y_i|X_{i-1}, \dots, X_0] \leq 0$ . W.l.o.g., assume that  $\mathbb{E}[Y_i|X_{i-1}, \dots, X_0] = 0$ , otherwise we can increase the values of negative  $Y_i$ 's which only increases the objective function  $\mathbb{E}[e^{tY_i}]$ . Note that  $\mathbb{E}[Y_i] = 0$ ,  $Y_i \leq c$  and  $Y_i$  being uniform over  $k$  values imply that  $|Y_i| \leq c(k-1)$ . Let

$$h(y) = \frac{e^{tc(k-1)} + e^{-tc(k-1)}}{2} + \frac{e^{tc(k-1)} - e^{-tc(k-1)}}{2} \cdot \frac{y}{c(k-1)}$$

be the line going through points  $(-c(k-1), e^{-tc(k-1)})$  and  $(c(k-1), e^{tc(k-1)})$ . From convexity of  $e^{tY}$ ,  $e^{tY} \leq h(y)$  for  $|y| \leq c(k-1)$ . Thus,

$$\mathbb{E}[e^{tY_i}] \leq \mathbb{E}[h(Y_i)] = h(\mathbb{E}[Y_i]) = h(0) = \cosh(tc(k-1)) \leq \exp(t^2 c^2 (k-1)^2 / 2),$$

where the last inequality  $\cosh(x) \leq \exp(x^2/2)$  for  $x > 0$  can be proven by comparing the Taylor series of the two functions.

Now,

$$\begin{aligned} \mathbb{E}\left[\exp\left(t \cdot \sum_{i=1}^m Y_i\right)\right] &= \mathbb{E}\left[\exp\left(t \cdot \sum_{i=1}^{m-1} Y_i\right) \cdot \mathbb{E}[\exp(t \cdot Y_m) | X_{m-1}, \dots, X_0]\right] \\ &\leq \mathbb{E}\left[\exp\left(t \cdot \sum_{i=1}^{m-1} Y_i\right)\right] \cdot \exp(t^2 c^2 (k-1)^2 / 2) \leq \exp(mt^2 c^2 (k-1)^2 / 2), \end{aligned}$$

which for  $t = \lambda / mc^2(k-1)^2$  implies  $\Pr[X_m - X_0 \geq \lambda] \leq \exp\left(\frac{-\lambda^2}{2mc^2(k-1)^2}\right)$ . □

Note that we have an extra factor of  $(k - 1)^2$  comparing with the normal form of Azuma's inequality, but we do not assume that  $X_i - X_{i-1}$  is bounded from below.

### 3 Toolkit

#### 3.1 Main Theorem

In this subsection we prove the main technical theorem that allows us to get circuit complexity lower bounds and #SAT upper bounds.

**Definition 3.1.** Let  $\{v_1, \dots, v_m\}$  be splitting vectors, and each  $v_i$  is a splitting vector of length  $2^{t_i} \geq 2$ . For a class of circuits  $\Omega$  (e.g.,  $\Omega = B_2$  or  $\Omega = U_2$ ), a set of substitutions  $\mathcal{S}$ , and a circuit complexity measure  $\mu$ , we write

$$\text{Splitting}(\Omega, \mathcal{S}, \mu) \preceq \{v_1, \dots, v_m\}$$

as a shortcut for the following statement: For any normalized circuit  $C$  from the class  $\Omega$  one can find in time  $\text{poly}(|C|)$  either a substitution<sup>5</sup> from  $\mathcal{S}$  whose splitting vector with respect to  $\mu$  belongs to the set  $\{v_1, \dots, v_m\}$  or a substitution that trivializes the output gate of  $C$ . A substitution always trivializes at least one gate (in particular, when we assign a constant to a variable we trivialize an input gate) and eliminates at least one variable.

**Theorem 4.** If  $\text{Splitting}(\Omega, \mathcal{S}, \mu) \preceq \{v_1, \dots, v_m\}$  and the longest splitting vector has length  $2^k$ , then

1. There exists an algorithm solving #SAT for circuits over  $\Omega$  in time  $O^*(\gamma^{\mu(C)})$ , where

$$\gamma = \max_{i \in [m]} \{\tau(v_i)\}.$$

2. If  $f \in B_n$  is an  $(\mathcal{S}, n, r)$ -dispenser, then

$$\mu(f) \geq \beta_w \cdot (r - k + 1), \text{ where } \beta_w = \min_{i \in [m]} \{\bar{v}_{i_{\max}}\}.$$

3. If  $f \in B_n$  is an  $(\mathcal{S}, n, r, \varepsilon)$ -extractor, then for every  $\mu < \beta_a \cdot r$ ,

$$\mu(f, \delta) \geq \mu, \text{ where } \beta_a = \min_{i \in [m]} \{\bar{v}_{i_{\text{avg}}}\} \text{ and } \beta_m = \min_{i \in [m]} \{\bar{v}_{i_{\min}}\},$$

$$\delta = \varepsilon + \exp\left(\frac{-(r \cdot \beta_a - \mu)^2}{2r(\beta_a - \beta_m)^2(2^k - 1)^2}\right).$$

*Proof.* We present a proof for a special case when all splitting vectors have length 2 (i.e.,  $k = 1$ ):  $\{v_1, \dots, v_m\} = \{(a_1, b_1), \dots, (a_m, b_m)\}$ . This makes the exposition simpler, and it is easy to see that the general statement follows by the same argument. In this case,

$$\beta_w = \min_{i \in [m]} \{\max\{a_i, b_i\}\}, \quad \beta_a = \min_{i \in [m]} \left\{ \frac{a_i + b_i}{2} \right\}, \quad \beta_m = \min_{i \in [m]} \{\min\{a_i, b_i\}\}.$$

---

<sup>5</sup>Here we assume that the circuit obtained from  $C$  by the substitution and normalization belongs to  $\Omega$  too.

1. Consider the following branching algorithm for #SAT. We describe the algorithm as a branching tree, each node of which contains a Boolean circuit and a set of currently made substitutions. The root of the tree is  $(C, \emptyset)$  — the input circuit and an empty set of substitutions. The nodes where the circuit is trivialized are called leaves. At each internal node (a node that is not a leaf) the algorithm finds in polynomial time substitutions  $x_i \leftarrow s$  and  $x_i \leftarrow s \oplus 1$  guaranteed by the theorem statement. Then the algorithm recursively calls itself on two circuits obtained from the current one by substituting  $x_i \leftarrow s$  and  $x_i \leftarrow s \oplus 1$ . That is, the algorithm adds to the current node  $(C, S)$  two children  $(C|x_i \leftarrow s, S \cup \{x_i \leftarrow s\})$  and  $(C|x_i \leftarrow s \oplus 1, S \cup \{x_i \leftarrow s \oplus 1\})$ . Note that the statement guarantees that the substitutions  $x_i \leftarrow s$  and  $x_i \leftarrow s \oplus 1$  either give us an  $(a_i, b_i)$ -splitting for some  $i$  (i.e., decrease the measure  $\mu$  by at least  $a_i$  in one branch, and  $b_i$  in the other one), or trivialize the circuit and produce two leaves.

At each leaf the algorithm counts the number  $V$  of satisfying assignments: If the formula is constant zero, then  $V = 0$ , otherwise,  $V = 2^v$ , where  $v$  is the number of variables in the current formula. Indeed, for each assignment to the  $v$  variables, there exists a unique assignment to the rest of the variables (via the substitutions at the leaf), and the circuit remains constant 1. Since substitutions  $x_i \leftarrow s$  and  $x_i \leftarrow s \oplus 1$  lead to different assignments to the input variables, the leaves of the branching tree correspond to disjoint sets of assignments. Therefore, the total number of satisfying assignments of the original circuit is the sum of the number of satisfying assignments at the leaves of the tree. Since the running time of the algorithm at each node is polynomial, the total running time is bounded from above by  $O^*(\gamma^{\mu(C)})$ , where  $\gamma = \max_{i \in [m]} \{\tau(a_i, b_i)\}$ .

2. For every pair of integers  $(n, r)$  such that  $n \geq r \geq 0$ , let  $\mathcal{F}_{n,r} \subseteq B_n$  denote the class of functions from  $\{0, 1\}^n$  to  $\{0, 1\}$  that are not constant after any  $r$  substitutions from  $\mathcal{S}$ . We show that for every  $f \in \mathcal{F}_{n,r}$ ,  $\mu(f) \geq \beta_w \cdot r$ .

The proof of the claim proceeds by induction on  $r$ . For  $r = 0$  the statement is trivial. Now assume that  $r \geq 1$ . Consider substitutions  $x_i \leftarrow s$  and  $x_i \leftarrow s \oplus 1$  guaranteed by the theorem statement. Now select a value of  $c \in \{0, 1\}$  in such a way that the substitution  $x_i \leftarrow s \oplus c$  reduces the measure by at least  $\beta_w$ . Consider the function  $g$  of  $n - 1$  variables which is  $f$  restricted to  $x_i \leftarrow s \oplus c$ . By the theorem statement,  $\mu(f) \geq \beta_w + \mu(g)$ , and by the induction hypothesis,  $\mu(g) \geq \beta_w \cdot (r - 1)$ . Therefore,  $\mu(f) \geq \beta_w \cdot r$ .

3. Let us consider a circuit  $C$  such that  $\mu(C) \leq \beta_a \cdot r$ . Consider the branching tree from the 1st part of the proof. At each node of the branching tree let us uniformly at random choose a child we proceed to. Let  $\delta_i$  be the random variable that equals to the measure decrease at  $i$ th step ( $i$ th level of the branching tree, where 0 corresponds to the root). For  $i \geq 0$ , define the random variable

$$X_i = (i + 1) \cdot \beta_a - \sum_{j=0}^i \delta_j.$$

Let us show that the variable  $X_i$  is a supermartingale:

$$\begin{aligned} \mathbb{E}[X_i | X_{i-1}, \dots, X_0] &= i \cdot \beta_a - \sum_{j=0}^{i-1} \delta_j + (\beta_a - \mathbb{E}[\delta_i | X_{i-1}, \dots, X_0]) \\ &= X_{i-1} + (\beta_a - \mathbb{E}[\delta_i]) \leq X_{i-1}. \end{aligned}$$

Let  $Y_i = X_i - X_{i-1}$ . Then  $Y_i$  is distributed uniformly over at most 2 values, and  $Y_i \leq \beta_a - \beta_m$ . Now let  $\lambda = \beta_a \cdot r - \mu(C)$ . Then, by Lemma 3:

$$\Pr[X_r - X_0 \geq \lambda] \leq \exp\left(\frac{-\lambda^2}{2r(\beta_a - \beta_m)^2}\right).$$

Now we want to bound from above the correlation between  $f$  and the function given by the branching tree. Note that all leaves of the tree that have depth smaller than  $r$  altogether give correlation at most  $\varepsilon$  with the extractor  $f$  (since each of these leaves defines an  $(\mathcal{S}, n, r)$  source). Now let us count the number of nodes at the depth  $r$ . There are at most  $2^r$  possible nodes, but each of them survives till the  $r$ th level only with probability  $\Pr[X_r - X_0 \geq \lambda]$ . Indeed, if  $X_r - X_0 < \lambda$ , then  $\sum_{j=1}^r \delta_j > \mu(C)$ , which means that the function becomes constant before the  $r$ th level. Therefore, there are at most  $2^r \cdot \Pr[X_r - X_0 \geq \lambda]$  nodes at the depth  $r$ . Since each node at the depth  $r$  has  $r$  inputs fixed, it covers at most  $2^{n-r}$  points of the Boolean cube. Therefore, the total correlation is bounded from above by:

$$\text{Cor}(f, C) \leq \varepsilon + \exp\left(\frac{-\lambda^2}{2r(\beta_a - \beta_m)^2}\right) = \varepsilon + \exp\left(\frac{-(r \cdot \beta_a - \mu(C))^2}{2r(\beta_a - \beta_m)^2}\right).$$

□

### 3.2 Discussion

Many known lower bounds for circuits with unrestricted depth can be proved using this framework, in particular, the strongest known lower bounds over  $B_2$  and  $U_2$ . Schnorr [54] proved a  $2n - \Theta(1)$  on  $C_{B_2}$  for a wide class of functions using  $\mu(C) = s(C)$  and bit fixing substitutions. Stockmeyer [58] proved a  $2.5n - \Theta(1)$  lower bound for symmetric functions using  $\mu(C) = s(C)$  and a special case of projections:  $\{x_i \leftarrow c, \{x_i \leftarrow f, x_j \leftarrow f \oplus 1\}\}$  (the latter “double” substitution essentially fixes  $x_i \oplus x_j$  to 1; by applying such a substitution to, say, the majority function one gets the majority function of fewer inputs). Kojevnikov and Kulikov [35] improved the bound by Schnorr to  $7n/3 - \Theta(1)$  using the measure  $\mu(C) = 3x(C) + 2a(C)$  assigning different weights to xor-gates and and-gates. Demenkov and Kulikov [22] proved a  $3n - o(n)$  lower bound for an affine disperser for sublinear dimension using  $\mu(C) = s(C) + i(C)$  and affine substitutions. Recently, Find et al. [27] extended this approach to get a  $(3 + 1/86)n - o(n)$  lower bound for the same function using a few additional tricks (while the measure and the set of allowed substitutions are not easy to describe).

For the basis  $U_2$ , Schnorr [54] proved that the circuit size of parity is  $3n - 3$  using bit fixing substitutions. Zwick [68] proved a  $4n - \Theta(1)$  lower bound for symmetric functions using bit-fixing substitutions and  $\mu(C) = s(C) - i_1(C)$ . His measure was then used by Lachish and Raz [39] and by Iwama and Morizumi [33] to prove  $4.5n - o(n)$  and  $5n - o(n)$  lower bounds for strongly two-dependent functions. Recently, Demenkov et al. [23] gave a simpler proof of a  $5n - o(n)$  lower bound

for a linear function with  $o(n)$  outputs. All these proofs use bit fixing substitutions only, however the case analysis can be simplified using also projections and a measure of the form  $\mu = s + \alpha \cdot i$ .

At the same time, there are known lower bound proofs that use additional tricks. E.g., Blum [10] to prove a  $3n - o(n)$  lower bound over  $B_2$  first considers a few cases when it is easy to remove three gates, and for all the remaining circuits shows a lower bound directly by counting the number of gates using some particular properties of the function under consideration.

Also, upper bounds for SAT and #SAT for various circuits classes (and for many other NP-hard problems) are proved by making substitutions recursively and using a carefully chosen measure to estimate the complexity decrease after substitutions.

The whole framework is a formalization of the following simple idea. To prove a lower bound  $\zeta n$  on circuit size one usually shows that there always exists a substitution  $x_i \leftarrow f$  eliminating at least  $\zeta$  gates from the circuit. By analyzing also the complexity decrease under the substitution  $x_i \leftarrow f \oplus 1$  (i.e., the complementary branch) one gets an upper bound for #SAT and an average case lower bound.

Recently, Golovnev et al. [28] showed that this approach alone cannot lead to superlinear circuit lower bounds or subexponential #SAT upper bounds. They showed that there are circuits such that any substitution  $x_i \leftarrow f$  (where  $f$  is *any* function!) cannot eliminate more than five gates from it. They also generalized this to other natural circuit complexity measures and sequences of substitutions.

## 4 Bounds for the Basis $U_2$

In this and the following sections, we prove known and new circuit lower bounds and upper bounds for #SAT. As discussed already, the main ingredient of all proofs is the case analysis showing the existence of a substitution reducing the measure by a sufficient amount. Usually, in such proofs we argue as follows: take a gate  $A$  and make a substitution trivializing  $A$ ; this eliminates  $A$  and all its successors. However it might be the case that  $A$  is the output gate and so does not have any successors. This means that we are at the end of the gate elimination process or at the leaf of a recursion tree. This, in turn, means that we do not need to estimate the current measure decrease. For this reason, in all the proofs below we assume implicitly that if we trivialize a gate then it is not the output gate.

### 4.1 Bit Fixing Substitutions

We start with a well-known case analysis of a  $3n - 3$  lower bound for the parity function over  $U_2$  due to Schnorr [54]. Using this case analysis we reprove the bounds given recently by Chen and Kabanets [15] in our framework. The analysis is basically the same though the measure is slightly different. We provide these results mostly as a simple illustration of using the framework.

**Lemma 5.**  $\text{Splitting}(U_2, \{x_i \leftarrow c\}, s + \alpha i) \preceq \{(\alpha, 2\alpha), (3 + \alpha, 3 + \alpha), (2 + \alpha, 4 + \alpha)\}$ .

*Proof.* Let  $A$  be a top-gate (that is, a gate fed by two inputs) computing  $(x_i \oplus a)(x_j \oplus b) \oplus c$  where  $x_i, x_j$  are input variables and  $a, b, c \in \{0, 1\}$  are constants. If  $\text{out}(x_i) = 1$  we split on  $x_j$ . When  $x_j \leftarrow b$  the gate  $A$  trivializes and the resulting circuit becomes independent of  $x_i$ . This gives  $(\alpha, 2\alpha)$ . Assume now that  $\text{out}(x_i) \geq 2$ . Denote by  $B$  the other successor of  $x_i$  and let  $C, D$  be successors of  $A, B$ , respectively. Note that  $B \neq C$  since the circuit is normalized but it might be

the case that  $C = D$ . We then split on  $x_i$ . Both  $A$  and  $B$  trivialize in at least one of the branches and their successors are also eliminated. This gives us either  $(3 + \alpha, 3 + \alpha)$  or  $(2 + \alpha, 4 + \alpha)$ . (Note if  $A$  and  $B$  trivialize in the same branch and  $C = D$  then we counted  $C$  twice in the analysis above. However in this case  $C$  also trivializes so all its successors are also eliminated.)  $\square$

**Corollary 6.** 1. For any  $\epsilon > 0$  there exists  $\delta = \delta(\epsilon) > 0$  such that #SAT for circuits over  $U_2$  of size at most  $(3 - \epsilon)n$  can be solved in time  $(2 - \delta)^n$ .

2.  $C_{U_2}(x_1 \oplus \cdots \oplus x_n \oplus c) \geq 3n - 6$ .

3.  $C_{U_2}\left(x_1 \oplus \cdots \oplus x_n \oplus c, \exp\left(\frac{-(t-9)^2}{18(n-1)}\right)\right) \geq 3n - t$ . This, in particular, implies that  $\text{Cor}(x_1 \oplus \cdots \oplus x_n \oplus c, C)$  is negligible for any circuit  $C$  of size  $3n - \omega(\sqrt{n \log n})$ .

*Proof.* 1. First note that for large enough  $\alpha$ , we have  $\tau(\alpha, 2\alpha) < \tau(3 + \alpha, 3 + \alpha) = 2^{\frac{1}{3+\alpha}} < \tau(2 + \alpha, 4 + \alpha)$ . Let  $\gamma(\alpha) = \tau(2 + \alpha, 4 + \alpha) - 2^{\frac{1}{3+\alpha}}$ . By Lemma 2,  $\gamma(\alpha) = O(1/\alpha^3)$  holds. The running time of the algorithm is at most

$$\begin{aligned} (\tau(2 + \alpha, 4 + \alpha))^{s+\alpha n} &\leq \left(2^{\frac{1}{3+\alpha}}(1 + \gamma(\alpha))\right)^{s+\alpha n} \leq 2^{\frac{s+\alpha n}{3+\alpha}} 2^{(s+\alpha n)\gamma(\alpha) \log_2 e} \\ &\leq 2^{\frac{(3-\epsilon)n+\alpha n}{3+\alpha} + O(n/\alpha^2)} \leq (2 - \delta)^n \end{aligned}$$

for some  $\delta > 0$  if we set  $\alpha = c/\epsilon$  for large enough  $c > 0$ .

2. The parity function does not become constant under any  $n - 1$  substitutions of variables to constants. Lemma 5 guarantees that for  $\alpha = 3$  we can always assign a constant to a variable so that  $s + 3i$  is reduced by at least 6. Hence for any circuit  $C$  over  $U_2$  computing parity,  $s(C) + 3n \geq 6(n - 1)$  implying  $s(C) \geq 3n - 6$ . (In fact, it can be shown by a slightly more careful analysis that  $C_{U_2}(x_1 \oplus \cdots \oplus x_n) = 3n - 3$ .)

3. The parity function is  $(\{x_i \leftarrow c\}, n, n - 1, 0)$ -extractor. Let us consider a circuit  $C$  of size at most  $3n - t$ , that is,  $\mu(C) \leq (3n - t) + \alpha n$ . Now we fix  $\alpha = 6$ , then  $\beta_a = \min\{9, 9, 9\} = 9$ ,  $\beta_m = \min\{6, 9, 8\} = 6$ .

We use the third item of Theorem 4 with  $k = 1$ ,  $r = n - 1$ ,  $\varepsilon = 0$ ,  $\mu = (3n - t + 6n)$ , which gives us

$$\delta = \exp\left(\frac{-(9(n-1) - (3n-t+6n))^2}{18(n-1)}\right) = \exp\left(\frac{-(t-9)^2}{18(n-1)}\right).$$

$\square$

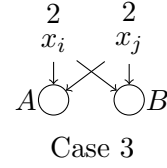
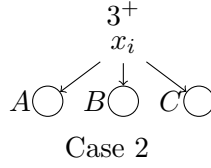
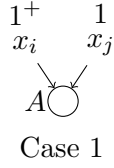
## 4.2 Projections

In this subsection, we prove new bounds for the basis  $U_2$ . The two main ideas leading to improved bounds are using projections to handle the Case 3 below and using 1-variables to get better estimates for complexity decrease (this trick was used by Zwick [68] and then by [39, 33]).

**Lemma 7.** For  $0 \leq \sigma \leq 1/2$ ,

$$\begin{aligned} \text{Splitting}(U_2, \{x_i \leftarrow c, x_i \leftarrow x_j \oplus c\}, s + \alpha i - \sigma i_1) &\preceq \\ &\{(\alpha, 2\alpha), (2\alpha, 2\alpha, 2\alpha, 3\alpha), (3 + \alpha + \sigma, 3 + \alpha), (4 + \alpha + \sigma, 2 + \alpha)\}. \end{aligned}$$

*Proof.* Note that for every eliminated gate we decrease the measure by at least  $1 - \sigma \geq 1/2$ , if some gate becomes constant we decrease the measure by at least 1.

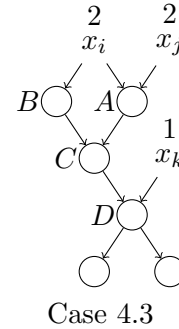
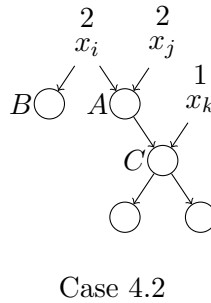
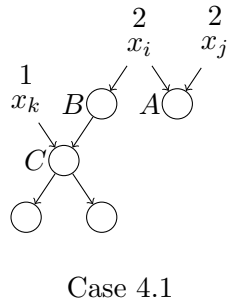


Case 1. There is a top gate  $A$  fed by 1-variable  $x_j$ . Assigning  $x_i$  a constant we trivialize the gate  $A$  in one of the branches and loose the dependence on  $x_j$ . Thus, we get at least  $(\alpha, 2\alpha)$  splitting.

Case 2. There is a variable  $x_i$  of degree at least 3. Neither of  $A, B, C$  is fed by a 1-variable otherwise we would be in the Case 1. When we assign  $x_i$  a constant, gates  $A, B$ , and  $C$  become constant in one of the branches. Hence in one of the branches we eliminate also at least one extra gate. Thus, we get at least  $(4 + \alpha - \sigma, 3 + \alpha)$  splitting vector which dominates  $(3 + \alpha + \sigma, 3 + \alpha)$  (since  $\sigma \leq 1/2$ ).

Case 3. There are two 2-variables that feed the same two top gates. Let the gates  $A$  and  $B$  compute Boolean functions  $f_A(x_i, x_j) = (x_i \oplus a_A)(x_j \oplus b_A) \oplus c_A$  and  $f_B(x_i, x_j) = (x_i \oplus a_B)(x_j \oplus b_B) \oplus c_B$  respectively. If  $a_A = a_B$  or  $b_A = b_B$  then we assign  $x_i \leftarrow a_A$  or  $x_j \leftarrow b_A$  respectively and make both gates constant. Otherwise,  $f_B(x_i, x_j) = (x_i \oplus a_A \oplus 1)(x_j \oplus b_A \oplus 1) \oplus c_B$ . It is easy to see that if  $x_i \oplus a_A \oplus x_j \oplus b_A = 1$  then both functions are constant. Hence, the substitution  $x_i \leftarrow a_A \oplus x_j \oplus b_A \oplus 1$  makes  $A$  and  $B$  constant as well. In both cases there is a substitution that makes  $A$  and  $B$  constant and therefore eliminates the dependence on  $x_j$ , so we get at least  $(\alpha, 2\alpha)$  splitting vector.

Case 4. There are three gates that are fed by two 2-variables. Let  $A$  be the common successor of  $x_i$  and  $x_j$ .



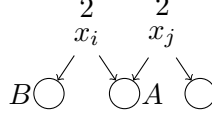
Case 4.1. Gate  $B$  is a 1-gate with a successor  $C$  that is a  $2^+$ -gate fed by the 1-variable  $x_k$ . If we assign constants to  $x_j$  and  $x_k$  we eliminate the dependence on  $x_i$  since both gates  $A$  and  $C$  are trivialized in one of the branches, so we get  $(2\alpha, 2\alpha, 2\alpha, 3\alpha)$  splitting.

Case 4.2. Gate  $A$  is a 1-gate with a successor  $C$  that is a  $2^+$ -gate fed by 1-variable  $x_k$ . Analogous to the previous case if we assign constants to  $x_i$  and  $x_j$  we eliminate the dependence on  $x_k$  in one of the branches, so we get again  $(2\alpha, 2\alpha, 2\alpha, 3\alpha)$  splitting.



Case 4.3. Gates  $A$  and  $B$  and its common successor  $C$  are 1-gates and the only successor of  $C$  is a 2-gate fed by 1-variable  $x_k$ . When we assign constants to  $x_k$  gate  $D$  becomes constant in one of the branches, hence, gates  $A$ ,  $B$ ,  $C$  become unnecessary and we lose the dependence on variable  $x_i$ . So we have at least  $(\alpha, 2\alpha)$  splitting.

Case 4.4. None of three previous cases apply.



Previous cases ruled out the possibility that  $A$  or  $B$  has the only successor that contributes only  $1 - \sigma$  to the measure decrease: we know that each of  $A$  and  $B$  is either a  $2^+$ -gate and its successors contribute at least  $2(1 - \sigma) \geq 1$  or a 1-gate with a successor which is not a  $2^+$ -gate fed by a 1-variable and it contributes at least 1. We also know, that when  $A$  and  $B$  are 1-gates with a common successor, this successor is not a  $2^+$ -gate fed by a 1-variable, and hence it contributes at least 1.

Therefore, if we assign  $x_i$  a constant each of  $A$  and  $B$  becomes constant in one of the branches, so the successors of  $A$  and  $B$  either contribute at least 1 in both branches or contribute at least 2 in one of the branches. In addition,  $x_j$  becomes a 1-variable in the branch where  $A$  trivializes. Thus, we get either  $(3 + \alpha + \sigma, 3 + \alpha)$  or  $(4 + \alpha + \sigma, 2 + \alpha)$ .

□

**Corollary 8.** 1. For any  $\epsilon > 0$  there exists  $\delta = \delta(\epsilon) > 0$  such that #SAT for circuits over  $U_2$  of size at most  $(3.25 - \epsilon)n$  can be solved in time  $(2 - \delta)^n$ .

2. Let  $f \in B_n$  be an  $(n, r(n) = n - \log^{O(1)}(n))$ -projections disperser from [41]. Then  $C_{U_2}(f) \geq 3.5n - \log^{O(1)}(n)$ .

3. Let  $f \in B_n$  be an  $(n, r(n) = n - \sqrt{n}, \epsilon(n) = 2^{-n^{\Omega(1)}})$ -projections extractor from [49]. Then  $C_{U_2}(f, \delta) \geq 3.25n - t$ , where  $\delta = 2^{-n^{\Omega(1)}} + \exp\left(\frac{-(t - 10.25\sqrt{n})^2}{190.125(n - \sqrt{n})}\right)$ . This, in particular, implies that  $\text{Cor}(f, C)$  is negligible for any circuit  $C$  of size  $3.25n - \omega(\sqrt{n \log n})$ .

*Proof.* 1. Let  $\sigma = 1/2$ . First note that for large enough  $\alpha$ , we have

$$\begin{aligned} \tau(\alpha, 2\alpha) &< \tau(2\alpha, 2\alpha, 2\alpha, 3\alpha) < \tau(3.25 + \alpha, 3.25 + \alpha) = 2^{\frac{1}{3.25 + \alpha}} \\ &< \tau(3.5 + \alpha, 3 + \alpha) < \tau(4.5 + \alpha, 2 + \alpha). \end{aligned}$$

Let  $\gamma(\alpha) = \tau(4.5 + \alpha, 2 + \alpha) - 2^{\frac{1}{3.25 + \alpha}}$ . By Lemma 2,  $\gamma(\alpha) = O(1/\alpha^3)$  holds. The running time of the algorithm is at most

$$\begin{aligned} (\tau(4.5 + \alpha, 2 + \alpha))^{s + \alpha n} &\leq \left(2^{\frac{1}{3.25 + \alpha}} (1 + \gamma(\alpha))\right)^{s + \alpha n} \\ &\leq 2^{\frac{s + \alpha n}{3.25 + \alpha}} 2^{(s + \alpha n)\gamma(\alpha) \log_2 e} \\ &\leq 2^{\frac{(3.25 - \epsilon)n + \alpha n}{3.25 + \alpha} + O(n/\alpha^2)} \leq (2 - \delta)^n \end{aligned}$$

for some  $\delta > 0$  if we set  $\alpha = c/\epsilon$  for large enough  $c > 0$ .

2. Lemma 7 guarantees that for  $\alpha = 7$ ,  $\sigma = 0.5$  one can always make an affine substitution reducing  $s + 7i$  by at least 10.5. The function  $f$  is resistant to  $r(n)$  such substitutions. Hence for a circuit  $C$  computing  $f$ ,  $s(C) + 7n \geq 10.5r(n)$ .
3. Let us consider a circuit  $C$  of size at most  $3.25n - t$ , that is,  $\mu(C) \leq (3.25n - t) + \alpha n$ . Now we fix  $\alpha = 7$ ,  $\sigma = 0.5$ , then

$$\beta_a = \min\{10.5, 15.75, 10.25, 10.25\} = 10.25, \quad \beta_m = \min\{7, 7, 10, 9\} = 7.$$

We use the third item of Theorem 4 with  $k = 2$ ,  $r = n - \sqrt{n}$ ,  $\varepsilon = 2^{-n^{\Omega(1)}}$ ,  $\mu = (3.25n - t + 7n)$ , which gives us

$$\begin{aligned} \delta &= 2^{-n^{\Omega(1)}} + \exp\left(\frac{-(10.25(n - \sqrt{n}) - (10.25n - t))^2}{2(n - \sqrt{n}) \cdot 3.25^2 \cdot 3^2}\right) \\ &= 2^{-n^{\Omega(1)}} + \exp\left(\frac{-(t - 10.25\sqrt{n})^2}{190.125(n - \sqrt{n})}\right). \end{aligned}$$

□

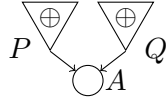
## 5 Bounds for the Basis $B_2$

### 5.1 Affine Substitutions

Here, we again start by reproving the bounds for  $B_2$  by Chen and Kabanets [15] by using the case analysis by Demenkov and Kulikov [22].

**Lemma 9.** *Splitting* $(B_2, \{x_i \leftarrow \bigoplus_{j \in J} x_j \oplus c\}, \mu = s + \alpha i) \preceq \{(\alpha, 2\alpha), (2 + \alpha, 3 + \alpha)\}$ .

*Proof.* Fix any topological ordering of a given circuit  $C$  and let  $A$  be the first gate in this ordering which is not a 1-xor (if there is no such gate then all the gates in  $C$  are 1-xors hence  $C$  computes an affine function and we can trivialize it with a single affine substitution). Note that the subcircuit underneath the gate  $A$  is a tree of xors, that is, a subcircuit consisting of 1-xors only. Let  $P$  and  $Q$  be inputs to  $A$ . Each of  $P$  and  $Q$  is computed by a tree of xors. Since each gate in such a tree has outdegree 1, it is not used in any other part of the circuit. Also, both  $P$  and  $Q$  might as well be input gates.



In any case, we can trivialize, say,  $P$  by an affine substitution. If  $P$  is an input gate this can be done simply by assigning the corresponding variable a constant. If  $P$  is an internal gate then it computes a sum  $\bigoplus_{j \in J} x_j \oplus c$ . To trivialize it, we select any variable  $i \in J$  and make a substitution  $x_i \leftarrow \bigoplus_{j \in J \setminus \{i\}} x_j \oplus c'$ . This clearly makes  $P$  constant and we can remove the whole tree computing  $P$  from the circuit (at this point, we use essentially the fact that all the gates in the tree computing  $P$  were needed to compute  $P$  only; hence when  $P$  is trivialized all these gates may be removed safely). If there are still some gates that depend on  $x_i$  then we build a new tree computing  $\bigoplus_{j \in J \setminus \{i\}} x_j \oplus c'$  and redirect all wires to  $x_i$  to this new tree. The new tree has one gate less than the removed one. So when  $P$  is an internal gate, by trivializing it we eliminate a variable and at least one gate.

Case 1.  $A$  is a  $2^+$ -xor. Then  $A$  itself is computed by a tree of 1-xors. Trivializing it gives  $(3 + \alpha, 3 + \alpha)$ .

Case 2.  $A$  is an and-gate and one of its inputs (say,  $P$ ) is an internal gate. We trivialize  $P$ . In both branches we eliminate  $A$  and  $P$ , but in one of them  $A$  is trivialized so we eliminate also its successors. This gives  $(2 + \alpha, 3 + \alpha)$ .

Case 3.  $A$  is an and-gate fed by two variables  $x_i$  and  $x_j$ .

Case 3.1. The outdegree of one of them (say,  $x_i$ ) is at least 2. Then splitting on  $x_i$  gives  $(2 + \alpha, 3 + \alpha)$ .

Case 3.2.  $\text{out}(x_i) = \text{out}(x_j) = 1$ . Then splitting on  $x_i$  is  $(\alpha, 2\alpha)$ .

□

**Corollary 10.** 1. For any  $\epsilon > 0$  there exists  $\delta = \delta(\epsilon) > 0$  such that #SAT for circuits over  $B_2$  of size at most  $(2.5 - \epsilon)n$  can be solved in time  $(2 - \delta)^n$ .

2. Let  $f \in B_n$  be an  $(n, r(n) = n - \log^{O(1)}(n))$ -affine disperser from [41]. Then  $C_{B_2}(f) \geq 3n - \log^{O(1)}(n)$ .

3. Let  $f \in B_n$  be an  $(n, r(n) = n - O(n/\log \log n), \varepsilon(n) = 2^{-n^{\Omega(1)}})$ -affine extractor from [40]. Then  $C_{B_2}(f, \delta) \geq 2.5n - t$ , where  $\delta = 2^{-n^{\Omega(1)}} + \exp\left(\frac{-(t - O(n/\log \log n))^2}{O(n)}\right)$ . This, in particular, implies that  $\text{Cor}(f, C)$  is negligible for any circuit  $C$  of size  $2.5n - \omega(n/\log \log n)$ .

*Proof.* 1. First note that for large enough  $\alpha$ , we have

$$\tau(\alpha, 2\alpha) < \tau(2.5 + \alpha, 2.5 + \alpha) = 2^{\frac{1}{2.5+\alpha}} < \tau(2 + \alpha, 3 + \alpha).$$

Let  $\gamma(\alpha) = \tau(2 + \alpha, 3 + \alpha) - 2^{\frac{1}{2.5+\alpha}}$ . By Lemma 2,  $\gamma(\alpha) = O(1/\alpha^3)$  holds. The running time of the algorithm is at most

$$\begin{aligned} (\tau(2 + \alpha, 3 + \alpha))^{s+\alpha n} &\leq \left(2^{\frac{1}{2.5+\alpha}}(1 + \gamma(\alpha))\right)^{s+\alpha n} \leq 2^{\frac{s+\alpha n}{2.5+\alpha}} 2^{(s+\alpha n)\gamma(\alpha)\log_2 e} \\ &\leq 2^{\frac{(2.5-\epsilon)n+\alpha n}{2.5+\alpha} + O(n/\alpha^2)} \leq (2 - \delta)^n \end{aligned}$$

for some  $\delta > 0$  if we set  $\alpha = c/\epsilon$  for large enough  $c > 0$ .

2. Lemma 9 guarantees that for  $\alpha = 3$  one can always make an affine substitution reducing  $s + 3i$  by at least 6. The function  $f$  is resistant to  $r(n)$  such substitutions. Hence for a circuit  $C$  computing  $f$ ,  $s(C) + 3n \geq 6r(n)$ .

3. Let us consider a circuit  $C$  of size at most  $2.5n - t$ , that is,  $\mu(C) \leq (2.5n - t) + \alpha n$ . Now we fix  $\alpha = 5$ , then  $\beta_a = \min\{7.5, 7.5\} = 7.5$ ,  $\beta_m = \min\{5, 7\} = 5$ .

We use the third item of Theorem 4 with  $k = 1$ ,  $r = r(n)$ ,  $\varepsilon = \varepsilon(n)$ ,  $\mu = (2.5n - t + 5n)$ , which gives us

$$\delta = \varepsilon(n) + \exp\left(\frac{-(7.5r(n) - (7.5n - t))^2}{12.5r(n)}\right) = \varepsilon(n) + \exp\left(\frac{-(t - 7.5(n - r(n)))^2}{12.5r(n)}\right).$$

□

## 5.2 Quadratic Substitutions

**Lemma 11.** For  $0 \leq \sigma \leq 1/3 \leq \alpha$ ,

$$\text{Splitting}(B_2, \{x_i \leftarrow p: \deg(p) \leq 2\}, s + \alpha i - \sigma i_1) \preceq \\ \{(\alpha, 2\alpha), (2\alpha, 2\alpha, 2\alpha, 3\alpha), (3 + \alpha - 2\sigma, 3 + \alpha - 2\sigma), (3 + \alpha + \sigma, 2 + \alpha)\}.$$

*Proof.* Fix any topological order of a given circuit  $C$  and let  $A$  be the first gate in this ordering which is not a 1-xor (if there is no such gate then all the gates in  $C$  are 1-xors hence  $C$  computes an affine function and we can trivialize it with a single affine substitution, that is, a function of degree at most 1). Then each input of  $A$  is a tree of xors, that is, a subcircuit consisting of 1-xors only. When we do an affine substitution to some variable that feed an xor-tree, we rebuild the tree and reduce the number of gates in it by at least one (it is explained in details in the proof of Lemma 9).

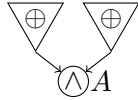


(In all the pictures of this proof we show only the type of the gates but not the actual functions computed at them.)

Case 1.  $A$  is a top and-gate fed by a 1-variable  $x_i$ . Similarly to the Case 1 of Lemma 7 we get  $(\alpha, 2\alpha)$ .

Case 2.  $A$  is  $2^+$ -xor. Let  $A$  compute  $c_A \oplus \bigoplus_{i \in I} x_i$  and  $I \subseteq \{1, \dots, n\}$ ,  $|I| \geq 2$ . If all  $x_i$ ,  $i \in I$ , are 1-variables then for any  $j \in I$  a substitution  $x_j \leftarrow c \oplus \bigoplus_{i \in I \setminus \{j\}} x_i$  eliminates the dependence on all 1-variables  $x_i$ ,  $i \in I \setminus \{j\}$ , so we get  $(2\alpha, 2\alpha)$ . Otherwise, there is at least one  $2^+$ -variable  $x_i$ ,  $i \in I$ . Substituting  $x_i \leftarrow c \oplus \bigoplus_{k \in I \setminus \{i\}} x_k$  we eliminate three gates in both branches, so we get at least  $(3 + \alpha - 2\sigma, 3 + \alpha - 2\sigma)$ .

Case 3.  $A$  is an and-gate which is not a top gate.

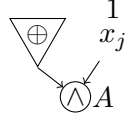


Let  $I, J \subset \{1, \dots, n\}$  be the sets of indices of variables in the left and in the right xor-trees respectively. W.l.o.g., we assume that  $|I| > 1$ , i.e. there is at least one gate in the left xor-tree feeding  $A$ .

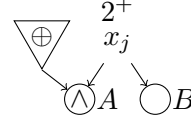
Case 3.1. There is a 1-variable  $x_i$  in the left tree. An affine substitution  $x_j \leftarrow c \oplus \bigoplus_{k \in J \setminus \{j\}} x_k$  for some  $j \in J$  eliminates two variables  $x_i$  and  $x_j$  in one of the branches, since the variable  $x_i$  becomes unnecessary in the branch where  $A$  becomes constant. The splitting is at least  $(\alpha, 2\alpha)$ .

Case 3.2. There is at least one gate in the right tree, i.e.  $|J| > 1$ . We apply an affine substitution  $x_i \leftarrow c \oplus \bigoplus_{k \in I \setminus \{i\}} x_k$  for some  $i \in I$  and eliminate four gates in one branch and two in the other one. Since  $x_i$  is a  $2^+$ -variable, this gives  $(4 + \alpha - \sigma, 2 + \alpha)$  which dominates  $(3 + \alpha + \sigma, 2 + \alpha)$ .

Case 3.3. The right tree consists of only one variable  $x_j$ , i.e.,  $|J| = 1$ .



Case 3.3.1



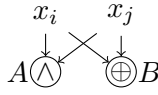
Case 3.3.2

Case 3.3.1.  $x_j$  is a 1-variable. An affine substitution  $x_i \leftarrow c \oplus \bigoplus_{k \in I \setminus \{i\}} x_k$  for some  $i \in I$  eliminates  $x_j$  in the branch where  $A$  becomes constant, so we get at least  $(\alpha, 2\alpha)$ .

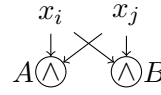
Case 3.3.2.  $x_j$  is a  $2^+$ -variable. Assigning  $x_j$  a constant we eliminate four gates in one branch and two in the other one. Gate  $B$  does not introduce new an 1-variable: if  $B$  is a  $2^+$ -gate fed by 1-variable we would be either in the Case 1 or in the Case 2. The splitting on  $x_j$  gives at least  $(4 + \alpha - \sigma, 2 + \alpha)$  which dominates  $(3 + \alpha + \sigma, 2 + \alpha)$ .

Case 4.  $A$  is a top and-gate fed by  $2^+$ -variables  $x_i$  and  $x_j$ .

Case 4.1. Both variables  $x_i$  and  $x_j$  feed the other gate  $B$ .



Case 4.1.1



Case 4.1.2

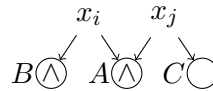
Case 4.1.1.  $B$  is an xor-gate. An affine substitution  $x_i \leftarrow x_j \oplus c$  eliminates the dependence on  $x_j$  in the branch where  $A$  becomes constant, so we get at least  $(\alpha, 2\alpha)$ .

Case 4.1.2.  $B$  is an and-gate. Similarly to the Case 3 of the proof of Lemma 7 we get  $(\alpha, 2\alpha)$ .

Case 4.2. Variables  $x_i$  and  $x_j$  feed three gates:  $A$ ,  $B$ , and  $C$ .

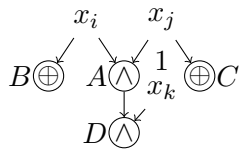
Note that in the following cases eliminating gates  $B$  and  $C$  we can not eliminate a 1-variable, otherwise we would be either in the Case 1 or in the Case 2.

Case 4.2.1. Gate  $B$  or  $C$  is an and-gate. W.l.o.g.,  $B$  is an and-gate.

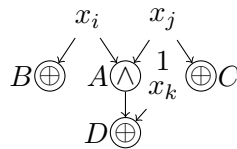


Assigning a constant to  $x_i$  we trivialize gates  $A$  and  $B$  in one of the branches, so we eliminate either four gates in one branch and two in the other one, or three gates in both branches. This gives either  $(4 + \alpha - 2\sigma, 2 + \alpha)$  or  $(3 + \alpha - \sigma, 3 + \alpha - \sigma)$ , which dominate  $(3 + \alpha + \sigma, 2 + \alpha)$  and  $(3 + \alpha - 2\sigma, 3 + \alpha - 2\sigma)$  respectively.

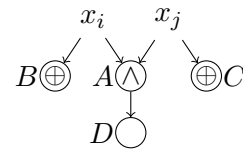
Case 4.2.2. Both  $B$  and  $C$  are xor-gates.



Case 4.2.2.1



Case 4.2.2.2



Case 4.2.2.3

Case 4.2.2.1.  $A$  is a 1-gate and its only successor  $D$  is an and-gate fed by the 1-variable  $x_k$ . Assigning constants to  $x_i$  and  $x_j$  we eliminate also the dependence on  $x_k$  in one of the branches. We get at least  $(2\alpha, 2\alpha, 2\alpha, 3\alpha)$ .

Case 4.2.2.2.  $A$  is a 1-gate and its only successor  $D$  is an xor-gate fed by the 1-variable  $x_k$ . Let gate  $A$  computes function  $(x_i \oplus a_A)(x_j \oplus b_A) \oplus c_A$ . A substitution  $x_k \leftarrow (x_i \oplus a_A)(x_j \oplus b_A) \oplus c$  of degree 2 makes  $D$  constant and eliminates at least three gates in each branch. So, we get at least  $(3 + \alpha - \sigma, 3 + \alpha - \sigma)$  which dominates  $(3 + \alpha - 2\sigma, 3 + \alpha - 2\sigma)$ . Note that  $x_k$  only feeds gate  $D$  which is now constant so we do not need to replace  $x_k$  by a subcircuit computing  $(x_i \oplus a_A)(x_j \oplus b_A) \oplus c$ .

Case 4.2.2.3.  $A$  is either a  $2^+$ -gate or a 1-gate with the only successor  $D$  which is not fed by a 1-variable. W.l.o.g., assume that outdegree of  $x_j$  is not greater than outdegree of  $x_i$ . Assigning  $x_i$  a constant we eliminate at least three gates in one branch and at least two in the other one. In addition either  $x_j$  becomes a 1-variable in the branch where  $A$  becomes constant (if  $x_j$  is a 2-variable) or  $x_i$  feeds more than 2 gates and we eliminate at least one more gate in both branches. Note that gate  $D$  does not introduce new 1-variables, otherwise we would be in one of the previous two cases. Thus, we get either at least  $(3 + \alpha + \sigma, 2 + \alpha)$  or even better splitting  $(4 + \alpha, 3 + \alpha - \sigma)$ .

□

**Corollary 12.** 1. For any  $\epsilon > 0$  there exists  $\delta = \delta(\epsilon) > 0$  such that #SAT for circuits over  $B_2$  of size at most  $(2.6 - \epsilon)n$  can be solved in time  $(2 - \delta)^n$ .

2. Let  $f \in B_n$  be an  $(n, r(n) = n - o(n))$ -quadratic disperser. Then  $C_{B_2}(f) \geq 3n - o(n)$ .

3. Let  $f \in B_n$  be an  $(n, r(n) = n - o(n), \varepsilon(n) = 2^{-\omega(\log n)})$ -quadratic extractor. Then  $C_{B_2}(f, \delta) \geq 2.6n - t$ , where  $\delta = 2^{-n^{\Omega(1)}} + \exp\left(\frac{-(t - 7.8(n - r(n)))^2}{121.68r(n)}\right)$ . This, in particular, implies that  $\text{Cor}(f, C)$  is negligible for any circuit  $C$  of size  $2.6n - g(n)$  for some  $g(n) = o(n)$ .

*Proof.* 1. Let  $\sigma = 1/5$ . First note that for large enough  $\alpha$ , we have

$$\tau(\alpha, 2\alpha) < \tau(2\alpha, 2\alpha, 2\alpha, 3\alpha) < \tau(2.6 + \alpha, 2.6 + \alpha) = 2^{\frac{1}{2.6 + \alpha}} < \tau(3.2 + \alpha, 2 + \alpha).$$

Let  $\gamma(\alpha) = \tau(3.2 + \alpha, 2 + \alpha) - 2^{\frac{1}{2.6 + \alpha}}$ . By Lemma 2,  $\gamma(\alpha) = O(1/\alpha^3)$  holds. The running time of the algorithm is at most

$$\begin{aligned} (\tau(3.2 + \alpha, 2 + \alpha))^{s + \alpha n} &\leq \left(2^{\frac{1}{2.6 + \alpha}} (1 + \gamma(\alpha))\right)^{s + \alpha n} \\ &\leq 2^{\frac{s + \alpha n}{2.6 + \alpha}} 2^{(s + \alpha n)\gamma(\alpha) \log_2 e} \\ &\leq 2^{\frac{(2.6 - \epsilon)n + \alpha n}{2.6 + \alpha} + O(n/\alpha^2)} \leq (2 - \delta)^n \end{aligned}$$

for some  $\delta > 0$  if we set  $\alpha = c/\epsilon$  for large enough  $c > 0$ .

2. Lemma 11 guarantees that for  $\alpha = 6$ ,  $\sigma = 0$  one can always make an affine substitution reducing  $s + 6i$  by at least 9. The function  $f$  is resistant to  $r(n)$  such substitutions. Hence for a circuit  $C$  computing  $f$ ,  $s(C) + 6n \geq 9r(n)$ .

3. Let us consider a circuit  $C$  of size at most  $2.6n - t$ , that is,  $\mu(C) \leq (2.6n - t) + \alpha n$ . Now we fix  $\alpha = 5.2$ ,  $\sigma = 0.2$ , then

$$\beta_a = \min\{7.8, 11.7, 7.8, 7.8\} = 7.8, \quad \beta_m = \min\{5.2, 5.2, 7.8, 7.2\} = 5.2.$$

We use the third item of Theorem 4 with  $k = 2$ ,  $r = r(n)$ ,  $\varepsilon = \varepsilon(n)$ ,  $\mu = (2.6n - t + 5.2n)$ , which gives us

$$\delta = \varepsilon(n) + \exp\left(\frac{-(7.8r(n) - (7.8n - t))^2}{2r(n) \cdot 2.6^2 \cdot 3^2}\right) = \varepsilon(n) + \exp\left(\frac{-(t - 7.8(n - r(n)))^2}{121.68r(n)}\right).$$

□

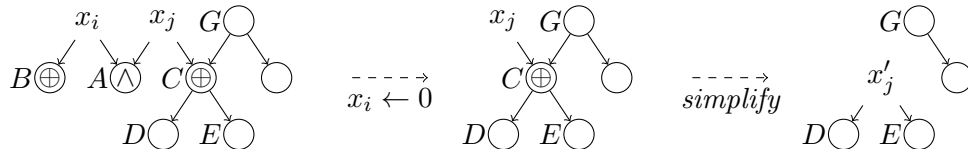
**Remark 1.** Note that it is an open problem to find an explicit construction of quadratic disperser or extractor over  $\mathbb{F}_2$  with  $r = n - o(n)$ . Any disperser for a slightly more general definition of quadratic varieties would also imply a new worst case lower bound [29].

**Remark 2.** Note that the upper bound for  $\#SAT$  can be improved using the following “forbidden trick”, that is, a simplification rule that reduces the size of a circuit without changing the number of its satisfying assignments, but changes the function computed by the circuit.

In the proof of Lemma 11 set  $\sigma = 0$  (that is, do not account for 1-variables). The set of splitting vectors then turn into

$$\{(\alpha, 2\alpha), (2\alpha, 2\alpha, 2\alpha, 3\alpha), (3 + \alpha, 3 + \alpha), (3 + \alpha, 2 + \alpha)\}.$$

By inspecting all the cases, we see that the splitting vector  $(3 + \alpha, 2 + \alpha)$  only appears in the the Case 4.2.2. We can handle this case differently: split on  $x_i$ . When  $A$  is trivialized,  $x_j$  becomes a 1-variable feeding an xor-gate. It is not difficult to show that by replacing this gate with a new variable  $x'_j$  one gets a circuit with the same number of satisfying assignments.



This additional trick gives us the following set of splitting vectors:

$$\{(\alpha, 2\alpha), (2\alpha, 2\alpha, 2\alpha, 3\alpha), (3 + \alpha, 3 + \alpha), (4 + \alpha, 2 + \alpha)\}.$$

These splitting numbers give an algorithm solving  $\#SAT$  in  $(2 - \delta(\epsilon))^n$  for  $B_2$ -circuits of size at most  $(3 - \epsilon)n$  for  $\epsilon > 0$ .

Note that such a simplification rule does not fit into our framework since it changes the function computed by a circuit. It would be interesting to adjust the framework to allow such kind of simplifications (probably, by incorporating some new parameter to the measure).

## 6 Open Problems

There are two natural questions left open in this paper.

1. Give an explicit construction of quadratic dispersers (see Remark 1).
2. Adjust the framework to allow using natural simplification rules like replacing an xor gate fed by a 1-variable for both upper bounds for  $\#SAT$  and lower bounds for circuit size (see Remark 2).

## Acknowledgement

We would like to thank the anonymous reviewers whose careful reading helped us to fix a critical flaw in one the proofs as well as to make the paper more readable.

## References

- [1] J. Alman, T. M. Chan, and R. Williams. Polynomial representations of threshold functions and algorithmic applications. In *Proceedings of the 57th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 467–476, 2016.
- [2] N. Alon and J. H. Spencer. *The probabilistic method*. John Wiley & Sons, 2004.
- [3] K. Amano and A. Saito. A nonuniform circuit class with multilayer of threshold gates having super quasi polynomial size lower bounds against NEXP. In *Proceedings of the 9th International Conference on Language and Automata Theory and Applications (LATA)*, pages 461–472, 2015.
- [4] K. Amano and A. Saito. A satisfiability algorithm for some class of dense depth two threshold circuits. *IEICE Transactions*, 98-D(1):108–118, 2015.
- [5] K. Amano and J. Tarui. A well-mixed function with circuit complexity  $5n$ : Tightness of the Lachish-Raz-type bounds. *Theor. Comput. Sci.*, 412(18):1646–1651, 2011.
- [6] P. Beame, R. Impagliazzo, and S. Srinivasan. Approximating  $AC^0$  by small height decision trees and a deterministic algorithm for  $\#AC^0$  SAT. In *Proceedings of the 27th Conference on Computational Complexity (CCC)*, pages 117–125, 2012.
- [7] E. Ben-Sasson and A. Gabizon. Extractors for polynomial sources over fields of constant order and small characteristic. *Theory of Computing*, 9:665–683, 2013.
- [8] E. Ben-Sasson and S. Kopparty. Affine dispersers from subspace polynomials. *SIAM J. Comput.*, 41(4):880–914, 2012.
- [9] E. Ben-Sasson and E. Viola. Short PCPs with projection queries. In *Proceedings of the 41st International Colloquium on Automata, Languages, and Programming (ICALP), Part I*, pages 163–173, 2014.



- [10] N. Blum. A Boolean function requiring  $3n$  network size. *Theor. Comput. Sci.*, 28:337–345, 1984.
- [11] H. Buhrman, L. Fortnow, and T. Thierauf. Nonrelativizing separations. In *Proceedings of the 13th Annual IEEE Conference on Computational Complexity (CCC)*, pages 8–12, 1998.
- [12] J.-Y. Cai.  $S_2 \subseteq ZPP^{NP}$ . In *Proceedings 2001 IEEE International Conference on Cluster Computing*, pages 620–628, Oct 2001.
- [13] V. T. Chakaravorthy and S. Roy. Arthur and Merlin as oracles. *Computational Complexity*, 20(3):505–558, 2011.
- [14] R. Chen. Satisfiability algorithms and lower bounds for Boolean formulas over finite bases. In *Proceedings of the 40th International Symposium on Mathematical Foundations of Computer Science (MFCS)*, pages 223–234, 2015.
- [15] R. Chen and V. Kabanets. Correlation bounds and #SAT algorithms for small linear-size circuits. *Theor. Comput. Sci.*, 654:2–10, 2016.
- [16] R. Chen, V. Kabanets, A. Kolokolova, R. Shaltiel, and D. Zuckerman. Mining circuit lower bound proofs for meta-algorithms. *Computational Complexity*, 24(2):333–392, 2015.
- [17] R. Chen, V. Kabanets, and N. Saurabh. An improved deterministic #SAT algorithm for small De Morgan formulas. *Algorithmica*, 76(1):68–87, 2016.
- [18] R. Chen and R. Santhanam. Improved algorithms for sparse MAX-SAT and MAX- $k$ -CSP. In *Proceedings of the 18th International Conference on Theory and Applications of Satisfiability Testing (SAT)*, pages 33–45, 2015.
- [19] R. Chen, R. Santhanam, and S. Srinivasan. Average-case lower bounds and satisfiability algorithms for small threshold circuits. In *Proceedings of the 31th Conference on Computational Complexity (CCC)*, pages 1:1–1:35, 2016.
- [20] B. Chor, O. Goldreich, J. Håstad, J. Friedman, S. Rudich, and R. Smolensky. The bit extraction problem of  $t$ -resilient functions (preliminary version). In *Proceedings of the 26th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 396–407, 1985.
- [21] G. Cohen and I. Shinkar. The complexity of DNF of parities. In *Proceedings of the 7th Innovations in Theoretical Computer Science Conference (ITCS)*, pages 47–58, 2016.
- [22] E. Demenkov and A. S. Kulikov. An elementary proof of a  $3n - o(n)$  lower bound on the circuit complexity of affine dispersers. In *Proceedings of the 36th International Symposium on Mathematical Foundations of Computer Science (MFCS)*, pages 256–265, 2011.
- [23] E. Demenkov, A. S. Kulikov, O. Melanich, and I. Mihajlin. New lower bounds on circuit size of multi-output functions. *Theory of Computing Systems*, 56(4):630–642, 2015.
- [24] Y. Dodis. *Exposure-resilient cryptography*. PhD thesis, Massachusetts Institute of Technology, 2000.
- [25] Z. Dvir. Extractors for varieties. *Computational Complexity*, 21(4):515–572, 2012.

- [26] Z. Dvir, A. Gabizon, and A. Wigderson. Extractors and rank extractors for polynomial sources. *Computational Complexity*, 18(1):1–58, 2009.
- [27] M. G. Find, A. Golovnev, E. A. Hirsch, and A. S. Kulikov. A better-than- $3n$  lower bound for the circuit complexity of an explicit function. In *IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 89–98, 2016.
- [28] A. Golovnev, E. A. Hirsch, A. Knop, and A. S. Kulikov. On the limits of gate elimination. In *41st International Symposium on Mathematical Foundations of Computer Science (MFCS)*, pages 46:1–46:13, 2016.
- [29] A. Golovnev and A. S. Kulikov. Weighted gate elimination: Boolean dispersers for quadratic varieties imply improved circuit lower bounds. In *Proceedings of the 7th Innovations in Theoretical Computer Science Conference (ITCS)*, pages 405–411, 2016.
- [30] J. Håstad. On the correlation of parity and small-depth circuits. *SIAM J. Comput.*, 43(5):1699–1708, 2014.
- [31] R. Impagliazzo, W. Matthews, and R. Paturi. A satisfiability algorithm for  $AC^0$ . In *Proceedings of the 23rd Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 961–972, 2012.
- [32] R. Impagliazzo, R. Paturi, and S. Schneider. A satisfiability algorithm for sparse depth two threshold circuits. In *Proceedings of the 54th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 479–488, 2013.
- [33] K. Iwama and H. Morizumi. An explicit lower bound of  $5n - o(n)$  for Boolean circuits. In *Proceedings of the 27th International Symposium on Mathematical Foundations of Computer Science (MFCS)*, pages 353–364, 2002.
- [34] H. Jahanjou, E. Miles, and E. Viola. Local reductions. In *Proceedings of the 42nd International Colloquium on Automata, Languages, and Programming (ICALP), Part I*, pages 749–760, 2015.
- [35] A. Kojevnikov and A. S. Kulikov. Circuit complexity and multiplicative complexity of Boolean functions. In *Proceedings of the 6th Conference on Computability in Europe (CiE)*, pages 239–245, 2010.
- [36] I. Komargodski and R. Raz. Average-case lower bounds for formula size. In *Proceedings of the 45th Symposium on Theory of Computing (STOC)*, pages 171–180, 2013.
- [37] I. Komargodski, R. Raz, and A. Tal. Improved average-case lower bounds for De Morgan formula size: Matching worst-case lower bound. *SIAM J. Comput.*, 46(1):37–57, 2017.
- [38] O. Kullmann. Fundamentals of branching heuristics. In A. Biere, M. Heule, H. van Maaren, and T. Walsh, editors, *Handbook of Satisfiability*, volume 185 of *Frontiers in Artificial Intelligence and Applications*, pages 205–244. IOS Press, 2009.
- [39] O. Lachish and R. Raz. Explicit lower bound of  $4.5n - o(n)$  for Boolean circuits. In *Proceedings on 33rd Annual ACM Symposium on Theory of Computing (STOC)*, pages 399–408, 2001.

- [40] X. Li. A new approach to affine extractors and dispersers. In *Proceedings of the 26th Annual IEEE Conference on Computational Complexity (CCC)*, pages 137–147, 2011.
- [41] X. Li. Improved two-source extractors, and affine extractors for polylogarithmic entropy. In *IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 168–177, 2016.
- [42] D. Lokshtanov, R. Paturi, S. Tamaki, R. R. Williams, and H. Yu. Beating brute force for systems of polynomial equations over finite fields. In *Proceedings of the 28th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 2190–2202, 2017.
- [43] B. Maurey. Espaces de Banach: Construction de suites symetriques. *C.R. Acad. Sci. Paris Ser. A-B.*, 288:679–681, 1979.
- [44] A. Nagao, K. Seto, and J. Teruyama. A moderately exponential time algorithm for  $k$ -IBDD satisfiability. In *Proceedings of the 14th International Symposium, on Algorithms and Data Structures (WADS)*, pages 554–565, 2015.
- [45] S. Nurk. An  $O(2^{0.4058m})$  upper bound for circuit SAT. Technical report, PDMI, 2009.
- [46] I. C. Oliveira. Algorithms versus circuit lower bounds. *Electronic Colloquium on Computational Complexity (ECCC)*, TR13-117, 2013.
- [47] R. Paturi, M. E. Saks, and F. Zane. Exponential lower bounds for depth three boolean circuits. *Computational Complexity*, 9(1):1–15, 2000.
- [48] W. J. Paul. A  $2.5n$ -lower bound on the combinational complexity of Boolean functions. *SIAM J. Comput.*, 6(3):427–443, 1977.
- [49] A. Rao. Extractors for low-weight affine sources. In *Proceedings of the 24th Annual IEEE Conference on Computational Complexity (CCC)*, pages 95–101, 2009.
- [50] T. Sakai, K. Seto, S. Tamaki, and J. Teruyama. Bounded depth circuits with weighted symmetric gates: Satisfiability, lower bounds and compression. In *41st International Symposium on Mathematical Foundations of Computer Science (MFCS)*, pages 82:1–82:16, 2016.
- [51] R. Santhanam. Circuit lower bounds for Merlin-Arthur classes. *SIAM J. Comput.*, 39(3):1038–1061, 2009.
- [52] R. Santhanam. Fighting perebor: New and improved algorithms for formula and QBF satisfiability. In *Proceedings of the 51th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 183–192, 2010.
- [53] R. Santhanam. Ironic complicity: Satisfiability algorithms and circuit lower bounds. *Bulletin of the EATCS*, 106:31–52, 2012.
- [54] C. Schnorr. Zwei lineare untere schranken für die komplexität boolescher funktionen. *Computing*, 13(2):155–171, 1974.
- [55] K. Seto and S. Tamaki. A satisfiability algorithm and average-case hardness for formulas over the full binary basis. *Computational Complexity*, 22(2):245–274, 2013.

- [56] R. Shaltiel. Dispersers for affine sources with sub-polynomial entropy. In *Proceedings of the 52nd Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 247–256, 2011.
- [57] C. E. Shannon. The synthesis of two-terminal switching circuits. *The Bell System Technical Journal*, 28(1):59–98, 1949.
- [58] L. J. Stockmeyer. On the combinational complexity of certain symmetric Boolean functions. *Mathematical Systems Theory*, 10:323–336, 1977.
- [59] A. Tal. #SAT algorithms from shrinkage. *Electronic Colloquium on Computational Complexity (ECCC)*, TR15-114, 2015.
- [60] S. Tamaki. A satisfiability algorithm for depth two circuits with a sub-quadratic number of symmetric and threshold gates. *Electronic Colloquium on Computational Complexity (ECCC)*, TR16-100, 2016.
- [61] F. Wang. NEXP does not have non-uniform quasipolynomial-size ACC circuits of  $o(\log \log n)$  depth. In *Proceedings of the 8th Annual Conference on Theory and Applications of Models of Computation (TAMC)*, pages 164–170, 2011.
- [62] R. Williams. Improving exhaustive search implies superpolynomial lower bounds. *SIAM J. Comput.*, 42(3):1218–1244, 2013.
- [63] R. Williams. Algorithms for circuits and circuits for algorithms. In *Proceedings of the 29th Annual IEEE Conference on Computational Complexity (CCC)*, pages 248–261, 2014.
- [64] R. Williams. New algorithms and lower bounds for circuits with linear threshold gates. In *Proceedings of the 46th Symposium on Theory of Computing (STOC)*, pages 194–202, 2014.
- [65] R. Williams. Nonuniform ACC circuit lower bounds. *J. ACM*, 61(1):2, 2014.
- [66] R. Williams. Natural proofs versus derandomization. *SIAM J. Comput.*, 45(2):497–529, 2016.
- [67] A. Yehudayoff. Affine extractors over prime fields. *Combinatorica*, 31(2):245–256, 2011.
- [68] U. Zwick. A  $4n$  lower bound on the combinational complexity of certain symmetric Boolean functions over the basis of unate dyadic Boolean functions. *SIAM J. Comput.*, 20(3):499–505, 1991.