These lecture notes are based on a manuscript of a book on matrix rigidity by Chi-Ning Chou and Sasha Golovnev.

Chapter 1

Introduction

1.1 Definitions and examples

Lecture 1

One of the main questions in computational complexity is that of proving lower bounds on the size of Boolean circuits computing explicitly given functions. While most Boolean functions of n inputs require circuits of size $2^n/n$ [Sha49a, Lup59a], we can only prove small linear lower bounds for explicitly defined functions [LR01, IM02, Blu83, DK11, FGHK16]. [1]

The same question remains open for *linear* circuits computing *linear* Boolean functions. Since any linear function with one output can be computed by a circuit of size at most n, we study linear functions with n inputs and n outputs. A random linear map with n outputs requires circuits of size $n^2/\log n$ [Lup56], but the best known lower bound for an explicit linear map is only $3n - o(n)$ [Cha94a].

The notion of matrix rigidity was introduced by Valiant [Val77] as a tool for proving lower bounds against linear circuits. (A related notion of separability was introduced by Grigoriev [Gri76].)

We will use the following notation. A matrix A is called s-sparse, if the number of non-zero entries in A is at most s. We will use $I_n, 0_n$ and J_n to denote the identity matrix, zero matrix, and all-ones matrix of size $n \times n$. For a matrix $A \in \mathbb{F}^{n\times n}$, by $\|A\|_0$ we denote the number of non-zero entries in A.

Definition 1.1 (Rigidity)**.** *Let $\mathbb{F}$ be a field, $A \in \mathbb{F}^{n\times n}$ be a matrix, and $0 \le r \le n$. The rigidity of A over $\mathbb{F}$, denoted by $\mathcal{R}^{\mathbb{F}}_A(r)$, is the Hamming distance between A and the set of matrices of rank at most r. Formally,*

$$\mathcal{R}^{\mathbb{F}}_A(r) := \min_{\operatorname{rank}(A+C)\le r} \|C\|_0 \ .$$

In other words, a matrix A has rigidity $\mathcal{R}^{\mathbb{F}}_A(r) \ge s$ if and only if $A \in \mathbb{F}^{n\times n}$ *cannot* be written as a sum

$$A = S + L \ ,$$

where $S \in \mathbb{F}^{n\times n}$ is $(s-1)$-sparse matrix, and $L \in \mathbb{F}^{n\times n}$ is low-rank: $\operatorname{rank}(L) \le r$.

Valiant [Val77] proved that any linear map $A \in \mathbb{F}^{n\times n}$ computed by a linear circuit (over a field $\mathbb{F}$) of depth $O(\log n)$ and size $o(n \log\log n)$ has rigidity at most $\mathcal{R}^{\mathbb{F}}_A(\varepsilon n) \le n^{1+\delta}$ for every constant $\varepsilon, \delta > 0$. Therefore, an explicit matrix of higher rigidity would give us a super-linear lower bound against linear circuits of logarithmic depth. Despite more than 40 years of research, the problem of proving super-linear lower bounds for such circuits remains open.

Let us now see the rigidity of a few specific matrices.

- If $A \in \mathbb{F}^{n\times n}$ has rank $\operatorname{rank}(A) = k$ over the field $\mathbb{F}$, then $\mathcal{R}^{\mathbb{F}}_A(r) = 0$ for every $r \ge k$. Indeed, A can be written as a sum of A and 0_n, where $\operatorname{rank}(A) \le r$ and 0_n is 0-sparse. Similarly, an s-sparse matrix $A \in \mathbb{F}^{n\times n}$ has rigidity $\mathcal{R}^{\mathbb{F}}_A(r) \le s$ for any value of r.

[1] Here by explicit functions we mean functions computable in time polynomial in n. We will later discuss the notion of explicitness in greater detail.

- For any $0 \le r \le n$, $\mathcal{R}^{\mathbb{F}}_{I_n}(r) = n - r$. Indeed, if we change $n - r$ ones of I_n to zeros, then the resulting matrix has rank r, which implies that $\mathcal{R}^{\mathbb{F}}_{I_n}(r) \le n - r$. On the other hand, for any $(n-r)$-sparse matrix B, from subadditivity of rank,

$$\operatorname{rank}(I_n + B) \ge \operatorname{rank}(I_n) - \operatorname{rank}(B) \ge n - (n - r) = r ,$$

  which gives us that $\mathcal{R}^{\mathbb{F}}_{I_n}(r) \ge n - r$.

- Let n be a multiple of $2r$, and let $M_n \in \mathbb{F}^{n\times n}$ be a matrix consisting of matrices I_{2r} stacked together side by side:

$$M_n = \begin{pmatrix} I_{2r} & \cdots & I_{2r} \\ \vdots & \ddots & \vdots \\ I_{2r} & \cdots & I_{2r} \end{pmatrix} .$$

  We will show that this matrix has rigidity $\mathcal{R}^{\mathbb{F}}_{A}(r) = \frac{n^2}{4r}$.

  Theorem 1.2 ([Mid05])**.** *For any field $\mathbb{F}$, and any n divisible by $1 \le 2r \le n$,*

$$\mathcal{R}^{\mathbb{F}}_{M_n}(r) = \frac{n^2}{4r} .$$

  Proof of Theorem 1.2. M_n consists of $\frac{n^2}{4r^2}$ copies of the identity matrix I_{2r}. In order to drop the rank of A to r, the rank of each copy of I_{2r} must be dropped to r. From the previous example we know that in order to decrease the rank of I_{2r} to r, one needs to change at least r elements. Thus, $\frac{n^2}{4r^2} \cdot r = \frac{n^2}{4r}$ entries of M_n must be changed. Note that this bounds is tight, *i.e.*, $\mathcal{R}^{\mathbb{F}}_{M_n}(r) = \frac{n^2}{4r}$. □

The bound of Theorem 1.2 easily generalizes to all values $r \le n/2$ with a loss of a multiplicative factor of 2. This theorem was proven by Midrijānis [Mid05], and it gives a simple matrix with rigidity $\mathcal{R}^{\mathbb{F}}_{M_n}(r) \ge \frac{n^2}{8r}$.

We will see later that there exist matrices with much higher rigidity $\tilde{\Omega}\left((n-r)^2\right)$. Embarrassingly, the best known lower bound for an *explicit* matrix improves on the $\frac{n^2}{8r}$ bound only by a logarithmic factor.

1.2 Circuit Complexity

A circuit corresponds to a simple straight line program where every instruction performs a binary operation on two operands, each of which is either an input or the result of a previous instruction. The structure of this program is extremely simple: no loops, no conditional statements. Still, we know no functions in P (or even NP, or even E^{NP}) that requires even $3.1n$ binary instructions ("size") to compute on inputs of length n. This is in sharp contrast with the fact that it is easy to *non-constructively* find such functions: simple counting arguments show a random function on n variables has circuit size $\Omega(2^n/n)$ with probability $1 - o(1)$ [Sha49b, Lup59b].

For small-depth circuits we know several strong lower bounds. (Note that when working with circuits of constant depth, we do not pose bounds on the fan-ins of the gates.) Depth-2 circuits (after a simple normalization) are just CNFs or DNFs. It is easy to see that the parity function $\oplus_n$ of n inputs requires CNFs and DNFs of size $\Omega(2^n)$. For depth-d circuits, we know a lower bound of $2^{\Omega(n^{(1/(d-1))})}$ [Hås86, HJP93, PPZ97, Bop97, PPSZ05, MW17]. Thus, for depth $d = o(\log n/\log\log n)$ we have non-trivial lower bounds even if the fan-ins of the gates are unbounded. For circuits with fan-in 2, we known functions which cannot be computed by circuits of depth $1.99\log n$ [Nec66]. Thus, a problem on the frontier is

Problem 1.3. *Prove a lower bound of $10n$ against circuits of depth $10\log n$.*
More generally, a lower bound of $\omega(n)$ against circuits of depth $O(\log n)$.

Super-linear lower bounds are not known even for linear circuits, i.e., circuits consisting of only gates computing linear combinations of their two inputs. Note that every linear function with one output has a circuit of size $n-1$ (and depth $\log n$). For linear circuits, we consider *linear transformations*, multi-output functions of the form $f(x) = Ax$ where $A \in \mathbb{F}^{n\times n}$. For a random matrix $A \in \{0,1\}^{n\times n}$, the size of the smallest linear circuit computing Ax is $\Theta(n^2/\log n)$ [Lup56] with probability $1 - o(1)$, but for explicitly-constructed matrices the strongest known lower bound is $3n - o(n)$ [Cha94b]. This leads us to another problem on the frontier:

Problem 1.4. *Prove a lower bound of $\omega(n)$ against* linear *circuits of depth $O(\log n)$.*

Formally, Problem 1.3 and Problem 1.4 are incomparable, as in the linear case we study a weaker computational model (which makes it easier to prove lower bounds), but are limited to proving lower bounds for a smaller class of problem (which makes it harder to prove lower bounds).

1.3 Circuits and Rigidity

In this section, we will present a seminal result of Valiant [Val77] showing that rigid matrices require log-depth circuits of super-linear size. We start with the definition of linear circuits.

Definition 1.5 (Linear circuits)**.** *Let $\mathbb{F}$ be a field and $n \in \mathbb{N}$. A circuit C with n inputs and n outputs is a directed acyclic graph where n vertices have fan-in zero and are labeled by the inputs, all other vertices have fan-in two and are labeled with* affine functions *(over $\mathbb{F}$) of their two inputs, n of these vertices are labeled as outputs. For every fixed input, the value at each node is computed by applying the corresponding functions. Such a circuit C naturally defines a linear map $f\colon \mathbb{F}^n \to \mathbb{F}^n$, and the corresponding matrix $A \in \mathbb{F}^{n\times n}$ such that $f(x) = Ax$.*

The depth $d(C)$ of a circuit C is the length of the longest path in the circuit. The size $s(C)$ of C is defined as the number of vertices in C.

The following theorem shows a connection between lower bounds for linear circuits and matrix rigidity.

Theorem 1.6. *Let $\mathbb{F}$ be a field, and $A \in \mathbb{F}^{n\times n}$ be a family of matrices for $n \in \mathbb{N}$. If $\mathcal{R}^{\mathbb{F}}_A(\varepsilon n) > n^{1+\delta}$ for constant $\varepsilon, \delta > 0$, then any $O(\log n)$-depth linear circuit computing $x \to Ax$ must be of size $\Omega(n \cdot \log\log n)$.*

The proof of Theorem 1.6 repeatedly uses the following beautiful graph theoretic lemma due to Erdös, Graham, and Szemerédi [EGS76]: If G is a directed acyclic graph with s edges and of depth d, then there is a set of $s/\log d$ edges whose removal decreases the depth of G by a factor of two. We will follow the proof of this lemma from [Vio09].

Lemma 1.7 ([EGS76])**.** *Let G be an acyclic digraph with s edges and of depth $d = 2^k$. There exists a set of $s/\log d$ edges in G such that after their removal, the longest path in G has length at most $d/2$.*

Proof of Lemma 1.7. For ease of exposition, we follow [Vio09] and define a *depth function*. Let $G = (V, E)$ be an acyclic digraph. We say that $D : V \to \{0, 1, \ldots, d\}$ is a depth function for G if for any $(a, b) \in E$, $D(a) < D(b)$. It is not difficult to see that G has depth at most d if and only if there exists a depth function $D : V \to \{0, 1, \ldots, d - 1\}$ for G.

We start with G of depth at most $d = 2^k$, and its depth function $D : V \to \{0, 1, \ldots, 2^k\}$. Now, consider the following partition of E using the depth function D. For each $i \in [k]$, define

$$E_i = \{(a, b) \in E : \text{ the most significant bit where } D(a), D(b) \text{ differ is the } i^{\text{th}} \text{ bit}\}.$$

As $\{E_i\}_{i\in[k]}$ is a partition of E, by the averaging argument, there exists $i^* \in [k]$ such that

$$|E_{i^*}| \le \frac{|E|}{k} \le \frac{|E|}{\log d}\,.$$

Now, it suffices to show that the depth of $G' = (V, E')$, where $E' = E \backslash E_{i^*}$, is at most 2^{k-1}. This can be shown by exhibiting a depth function $D' : V \to \{0, 1, \ldots, 2^{k-1} - 1\}$ for G'. The following shows that we can take $D'(v)$ to be $D(v)$ without the $i^{*\text{th}}$ bit.

Consider an edge $(a, b) \in E'$. Since $(a, b) \in E$, $D(a) < D(b)$. In particular, there exists $i \in [k]$ such that the most significant bit where $D(a)$ and $D(b)$ differ is i. Since $(a, b) \in E'$, the edge (a, b) was not removed, so $i \ne i^*$. Therefore, after removing the bit i^*, this bit i is still the most significant bit where $D'(a)$ and $D'(b)$ differ. This implies that $D'(a) < D'(b)$, and that $D' : V \to \{0, 1, \ldots, 2^{k-1} - 1\}$ is a depth function for G'. ∎

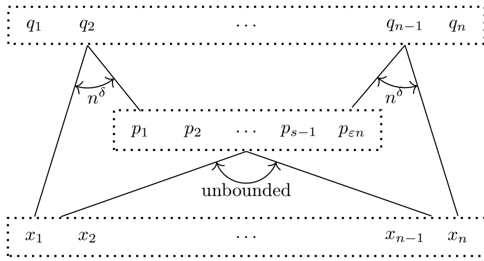

Figure 1.1: In order to compute the values of the outputs of the circuit C, first we precompute the values of εn removed edges (or vertices V'), and store them in variables p_i. Now each output q_j of the circuit C can be computed from n^δ inputs and precomputed bits. In particular, $A = BM + C$, where C encodes the dependence of the outputs q_i on the inputs x_j; B encodes the dependence of q_i on p_j; M encodes the dependence of p_i on x_j. Since C is sparse, and BM is low rank, the matrix A is not rigid.

Now we finish the proof of Theorem 1.6. **Lecture 2**

Proof of Theorem 1.6. We will show that for every constant $c_d \geq 2$, every circuit of depth at most $c_d \log n$ computing $x \to Ax$ must be of size at least $c_s n \log\log n$ for a constant $c_s = \frac{\varepsilon}{\log c_d + \log 1/\delta}$. Suppose, to the contrary, that there is a linear circuit C of size $s = c_s n \log\log n$ and depth $d = c_d \log n = 2^k$ that computes $x \to Ax$. Let G be the underlying acyclic digraph of C.

First, we apply Lemma 1.7 to G t times, and get a graph G' such that (i) only

$$s \cdot \left(\frac{1}{\log d} + \frac{1}{\log d - 1} + \cdots \frac{1}{\log d - (t-1)} \right) \leq \frac{st}{\log d - (t-1)}$$

edges are removed from G and (ii) the longest path in G' is of length at most $d' \leq d/2^t$.

By setting $t = \log c_d + \log 1/\delta$, the longest path in G' has length $\leq d/2^t = \delta \log n$, and the number of removed edges is at most

$$\frac{st}{\log d - (t-1)} = \frac{st}{\log d/2} \leq \frac{tc_s n \log\log n}{\log\log n} = \varepsilon n\,.$$

Now, let E be the set of removed edges and V' be the set of *tail* vertices of the edges from E. Since all paths in G' are no longer than d' and all in-degrees are at most 2, every output vertex in G' is now connected to at most $2^{d'}$ input variables. Therefore, every output is a (linear) function of at most $2^{d'}$ inputs and the functions computed at the removed edges (or the vertices V').

More specifically, let A_i be the i^{th} row of A, *i.e.,* the linear form computed by the i^{th} output vertex of G. Then A_i can be written as the following sum

$$A_i = \sum_{j \in [|V'|]} b_{ij} v_j + c_i$$

where v_j is the linear form computed by the j^{th} element in V' and c_i is the linear form computed by the i^{th} output vertex in G'. Note that since c_i only depends on at most $2^{d'}$ input variables.

Therefore, the matrix A can be written as follows.

$$A = BM + C$$

where $B \in \mathbb{F}^{n \times |V'|}$ consists of the coefficients b_{ij}, rows of $M \in \mathbb{F}^{|V'| \times n}$ compute linear forms of vertices from V', and $C \in \mathbb{F}^{n \times n}$ is a row sparse matrix where the number of non-zero entries in each row is at most $2^{d'} = n^\delta$.

The above argument gives us that $\tilde{\mathcal{R}}^{\mathbb{F}}_A(|V'|) = \tilde{\mathcal{R}}^{\mathbb{F}}_A(\varepsilon n) \leq n^\delta$, which contradicts the assumption on the rigidity of A.

□

1.4 Existence of Rigid Matrices

In this section, we will show that for any field $\mathbb{F}$, most of the $n \times n$ matrices have the highest possible rigidity for any rank parameter r.

It turns out that for every matrix A and field $\mathbb{F}$, there is a simple upper bound $\mathcal{R}_A^{\mathbb{F}}(r) \leq (n-r)^2$. Valiant [Val77] showed that this upper bound is essentially tight for a random matrix. First, we give a proof of the upper bound.

Theorem 1.8 (Simple upper bound)**.** *For any field $\mathbb{F}$, matrix $A \in \mathbb{F}^{n\times n}$, and integer $0 \leq r \leq n$, we have that*

$$\mathcal{R}_A^{\mathbb{F}}(r) \leq (n-r)^2 \ .$$

Proof of Theorem 1.8. If $\mathrm{rank}(A) \leq r$, then $\mathcal{R}_A^{\mathbb{F}}(r) = 0 \leq (n-r)^2$. Thus, it suffices to focus on the case where there is an $r \times r$ *full-rank* submatrix $B \in \mathbb{F}^{r\times r}$ of A. Without loss of generality, assume that B is located in the top left corner of A:

$$A = \begin{pmatrix} B & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \ , \tag{1.9}$$

where $A_{12} \in \mathbb{F}^{r\times(n-r)}$, $A_{21} \in \mathbb{F}^{(n-r)\times r}$, $A_{22} \in \mathbb{F}^{(n-r)\times(n-r)}$. In order to prove that $\mathcal{R}_A^{\mathbb{F}}(r) \leq (n-r)^2$, we will show that it is possible to change the entries in $A_{22} \in \mathbb{F}^{(n-r)\times(n-r)}$ and reduce the rank of A to r. Since B has full rank, each row in A_{21} is a *unique linear combination* of the rows in B. Thus, we can change the entries in A_{22} according to these linear combinations so that each row in A is now a linear combination of the first r rows, *i.e.,* the rank of the modified matrix is at most r. [2]

Note that the above algorithm only modifies the entries of $A_{22} \in \mathbb{F}^{(n-r)\times(n-r)}$. Thus, at most $(n-r)^2$ many entries in A are changed, and $\mathcal{R}_A^{\mathbb{F}}(r) \leq (n-r)^2$. □

We will now prove that almost all matrices have rigidity $(n-r)^2$.

Theorem 1.10 (Valiant's lower bounds [Val77])**.** *For any field $\mathbb{F}$,*

- *if $\mathbb{F}$ is infinite, then for all $0 \leq r \leq n$ there exists a matrix $M \in \mathbb{F}^{n\times n}$ of rigidity*

$$\mathcal{R}_M^{\mathbb{F}}(r) = (n-r)^2 \ ;$$

- *if $\mathbb{F}$ is finite, then for all $0 \leq r \leq n - \Omega(\sqrt{n})$ there exists a matrix $M \in \mathbb{F}^{n\times n}$ of rigidity*

$$\mathcal{R}_M^{\mathbb{F}}(r) = \Omega\left((n-r)^2/\log n\right) \ .$$

Proof of Theorem 1.10. Let $M_{r,s} = \{A \in \mathbb{F}^{n\times n} : \mathcal{R}_A^{\mathbb{F}}(r) \leq s\}$ be the set of all matrices of r-rigidity at most s. We will show that the n^2 elements of matrices from $M_{r,s}$ lie in the union of images of a few rational maps from $\mathbb{F}^{n^2+s-(n-r)^2}$ to $\mathbb{F}^{n^2}$. Intuitively, since for $s \ll (n-r)^2$ these images cover only a negligible fraction of all matrices in $\mathbb{F}^{n\times n}$, we will have that "most" of the matrices are rigid.

For every matrix $M \in M_{r,s}$, there exists an s-sparse matrix $S \in \mathbb{F}^{n\times n}$ and a low-rank matrix $L \in \mathbb{F}^{n\times n}, \mathrm{rank}(L) = k \leq r$ such that $M = S + L$. After one of at most $\binom{n}{k}^2$ permutations of rows and columns, we have the first k rows and columns of L linearly independent. The same permutations of rows and columns applied to M, give us a matrix of the form

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \ , \tag{1.11}$$

where $M_{11} \in \mathbb{F}^{k\times k}, M_{12} \in \mathbb{F}^{k\times(n-k)}, M_{21} \in \mathbb{F}^{(n-k)\times k}, M_{22} \in \mathbb{F}^{(n-k)\times(n-k)}$. Moreover, for at least one out of $\binom{n^2}{s}$ choices of s entries of the matrix, we have that a change in those entries makes $\mathrm{rank}(M_{11}) = \mathrm{rank}(M)$. Similarly to Theorem 1.8, this implies that all entries of M_{22} are then rational maps of the entries in

[2] Formally, we set $A_{22} = A_{21}B^{-1}A_{12}$.

M_{11}, M_{12}, M_{21}. That is, the n^2 entries of any matrix $M \in M_{r,s}$ lie in the union of at most $\binom{n}{r}^2 \cdot \binom{n^2}{s}$ rational maps from $\mathbb{F}^{s+n^2-(n-r)^2}$ to $\mathbb{F}^{n^2}$.

When $\mathbb{F}$ is infinite and $s < (n-r)^2$, every matrix in $M_{r,s}$ is in the union of finitely many images of rational functions from $\mathbb{F}^{n^2-1}$ to $\mathbb{F}^{n^2}$. Since n^2 rational functions of n^2-1 variables are algebraically dependent (see, e.g., [For92]), a finite union of such images is the set of roots of a non-zero polynomial. This implies that some matrices from $\mathbb{F}^{n\times n}$ do not belong to $M_{r,s}$.

When $|\mathbb{F}| = q < \infty$ is finite, each $M \in M_{r,s}$ is uniquely specified by one out of $\binom{n}{r}^2$ permutations, one of $\binom{n^2}{s}$ choices of s elements, values of those s elements, and values of the entries in M_{11}, M_{12}, M_{21}. Thus, the size of $M_{r,s}$ is bounded from above by

$$\binom{n}{r}^2 \cdot \binom{n^2}{s} \cdot q^s \cdot q^{n^2-(n-r)^2} \leq 2^{2n+2s\log n} \cdot q^{n^2+s-(n-r)^2},$$

which is at most $o(q^{n^2})$ for every $s < (n-r)^2/\Omega(\log_q n)$ and $r = n - \Omega(\sqrt{n})$. □

Note that the proof of the $\tilde{\Omega}\left((n-r)^2\right)$ lower bound in Theorem 1.10 does not provide a description of a rigid matrix, it merely proves its existence. This brings us to a discussion on explicitness of matrix constructions.

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