Problem 1 (Existence of Optimally Rigid Matrices)**.** Recall that in Theorem 1.10 we used the probabilistic method (counting argument) to prove that for all sufficiently large n there exist matrices $M \in \mathbb{F}_2^{n\times n}$ that have rigidity

$$\mathcal{R}_M^{\mathbb{F}_2}(\varepsilon n) \geq \Omega(n^2/\log n)\,.$$

In this exercise we will prove that there are even matrices $M \in \mathbb{F}_2^{n\times n}$ of rigidity

$$\mathcal{R}_M^{\mathbb{F}_2}(\varepsilon n) \geq \Omega(n^2)\,.$$

- Use an upper bound for binomial coefficients to improve the bound in Corollary 3.26 for linear rank $r = \Theta(n)$ and maximal sparsity $s = \Theta(n)$ from $2^{O(n^2 \log n)}$ to $2^{O(n^2)}$.
- Use this improved bound and the argument from the proof of Theorem 3.22 to conclude that a random $\Omega(n)$-regularly sparse matrix has rigidity $\Omega(n^2)$ for some rank parameter $r = \Omega(n)$.

Problem 2 (Rigidity in $\mathbf{E}$)**.** In class we showed that a rigid Hankel matrix [GT16] can be constructed in the complexity class $\mathbf{E}^{\mathbf{NP}}$. In this exercise we show that it can actually be constructed in $\mathbf{E}$.

- Examine the proof of Theorem 3.13 and conclude that it suffices to find a Hankel matrix whose k^2 submatrices have rigidity $\mathcal{R}_A(r) \geq \Omega\left(\frac{n}{\log n}\right)$.
- Give a deterministic algorithm running in time $2^{O(n)}$ that checks whether $\mathcal{R}_A^{\mathbb{F}_2}(r) \geq \Omega\left(\frac{n}{\log n}\right)$ for any matrix $A \in \mathbb{F}_2^{m\times m}$ and any $r, m \leq n$.

Problem 3 (Polynomial Method)**.** In class we proved that the number n of points in d dimensions whose pairwise (Euclidean) distances take only two values is upper bounded by $n \leq O(d^2)$.

- Use the same proof strategy to generalize this upper bound to sets with three pairwise distances. Let $x_1, \ldots, x_n \in \mathbb{R}^d$, and there exist numbers $d_1, d_2, d_3 > 0$ such that all pairwise distances $\|x_i - x_j\|^2 \in \{d_1, d_2, d_3\}$ for $i, j \in [n]$. Prove that $n \leq \mathsf{poly}(d)$. You do not need to optimize the upper bound, it would be sufficient to prove an upper bound of, say, $n \leq O(d^6)$.
- Construct a set $x_1, \ldots, x_n \in \mathbb{R}^d$ such that all pairwise distances take three values and $n \geq \Omega(d^3)$.
- *Optional.* Can you generalize the lower and upper bound to sets with k pairwise distances for arbitrarily large constant k?