

These lecture notes are based on a manuscript of a book on matrix rigidity by Chi-Ning Chou and Sasha Golovnev.

# Chapter 1

## Introduction

Lecture 1

### 1.1 Definitions and examples

One of the main questions in computational complexity is that of proving lower bounds on the size of Boolean circuits computing explicitly given functions. While most Boolean functions of  $n$  inputs require circuits of size  $2^n/n$  [Sha49a, Lup59a], we can only prove small linear lower bounds for explicitly defined functions [LR01, IM02, Blu83, DK11, FGHK16].<sup>1</sup>

The same question remains open for *linear* circuits computing *linear* Boolean functions. Since any linear function with one output can be computed by a circuit of size at most  $n$ , we study linear functions with  $n$  inputs and  $n$  outputs. A random linear map with  $n$  outputs requires circuits of size  $n^2/\log n$  [Lup56], but the best known lower bound for an explicit linear map is only  $3n - o(n)$  [Cha94a].

The notion of matrix rigidity was introduced by Valiant [Val77] as a tool for proving lower bounds against linear circuits. (A related notion of separability was introduced by Grigoriev [Gri76].)

We will use the following notation. A matrix  $A$  is called  $s$ -sparse, if the number of non-zero entries in  $A$  is at most  $s$ . We will use  $I_n, 0_n$  and  $J_n$  to denote the identity matrix, zero matrix, and all-ones matrix of size  $n \times n$ . For a matrix  $A \in \mathbb{F}^{n \times n}$ , by  $\|A\|_0$  we denote the number of non-zero entries in  $A$ .

**Definition 1.1** (Rigidity). *Let  $\mathbb{F}$  be a field,  $A \in \mathbb{F}^{n \times n}$  be a matrix, and  $0 \leq r \leq n$ . The rigidity of  $A$  over  $\mathbb{F}$ , denoted by  $\mathcal{R}_A^{\mathbb{F}}(r)$ , is the Hamming distance between  $A$  and the set of matrices of rank at most  $r$ . Formally,*

$$\mathcal{R}_A^{\mathbb{F}}(r) := \min_{\text{rank}(A+C) \leq r} \|C\|_0 .$$

In other words, a matrix  $A$  has rigidity  $\mathcal{R}_A^{\mathbb{F}}(r) \geq s$  if and only if  $A \in \mathbb{F}^{n \times n}$  *cannot* be written as a sum

$$A = S + L ,$$

where  $S \in \mathbb{F}^{n \times n}$  is  $(s-1)$ -sparse matrix, and  $L \in \mathbb{F}^{n \times n}$  is low-rank:  $\text{rank}(L) \leq r$ .

Valiant [Val77] proved that any linear map  $A \in \mathbb{F}^{n \times n}$  computed by a linear circuit (over a field  $\mathbb{F}$ ) of depth  $O(\log n)$  and size  $o(n \log \log n)$  has rigidity at most  $\mathcal{R}_A^{\mathbb{F}}(\varepsilon n) \leq n^{1+\delta}$  for every constant  $\varepsilon, \delta > 0$ . Therefore, an explicit matrix of higher rigidity would give us a super-linear lower bound against linear circuits of logarithmic depth. Despite more than 40 years of research, the problem of proving super-linear lower bounds for such circuits remains open.

Let us now see the rigidity of a few specific matrices.

- If  $A \in \mathbb{F}^{n \times n}$  has rank  $\text{rank}(A) = k$  over the field  $\mathbb{F}$ , then  $\mathcal{R}_A^{\mathbb{F}}(r) = 0$  for every  $r \geq k$ . Indeed,  $A$  can be written as a sum of  $A$  and  $0_n$ , where  $\text{rank}(A) \leq r$  and  $0_n$  is 0-sparse. Similarly, an  $s$ -sparse matrix  $A \in \mathbb{F}^{n \times n}$  has rigidity  $\mathcal{R}_A^{\mathbb{F}}(r) \leq s$  for any value of  $r$ .

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<sup>1</sup>Here by explicit functions we mean functions computable in time polynomial in  $n$ . We will later discuss the notion of explicitness in greater detail.

- For any  $0 \leq r \leq n$ ,  $\mathcal{R}_{I_n}^{\mathbb{F}}(r) = n - r$ . Indeed, if we change  $n - r$  ones of  $I_n$  to zeros, then the resulting matrix has rank  $r$ , which implies that  $\mathcal{R}_{I_n}^{\mathbb{F}}(r) \leq n - r$ . On the other hand, for any  $(n - r)$ -sparse matrix  $B$ , from subadditivity of rank,

$$\text{rank}(I_n + B) \geq \text{rank}(I_n) - \text{rank}(B) \geq n - (n - r) = r,$$

which gives us that  $\mathcal{R}_{I_n}^{\mathbb{F}}(r) \geq n - r$ .

- Let  $n$  be a multiple of  $2r$ , and let  $M_n \in \mathbb{F}^{n \times n}$  be a matrix consisting of matrices  $I_{2r}$  stacked together side by side:

$$M_n = \begin{pmatrix} I_{2r} & \cdots & I_{2r} \\ \vdots & \ddots & \vdots \\ I_{2r} & \cdots & I_{2r} \end{pmatrix}.$$

We will show that this matrix has rigidity  $\mathcal{R}_A^{\mathbb{F}}(r) = \frac{n^2}{4r}$ .

**Theorem 1.2** ([Mid05]). *For any field  $\mathbb{F}$ , and any  $n$  divisible by  $1 \leq 2r \leq n$ ,*

$$\mathcal{R}_{M_n}^{\mathbb{F}}(r) = \frac{n^2}{4r}.$$

*Proof of Theorem 1.2.*  $M_n$  consists of  $\frac{n^2}{4r^2}$  copies of the identity matrix  $I_{2r}$ . In order to drop the rank of  $A$  to  $r$ , the rank of each copy of  $I_{2r}$  must be dropped to  $r$ . From the previous example we know that in order to decrease the rank of  $I_{2r}$  to  $r$ , one needs to change at least  $r$  elements. Thus,  $\frac{n^2}{4r^2} \cdot r = \frac{n^2}{4r}$  entries of  $M_n$  must be changed. Note that this bound is tight, *i.e.*,  $\mathcal{R}_{M_n}^{\mathbb{F}}(r) = \frac{n^2}{4r}$ .  $\square$

The bound of Theorem 1.2 easily generalizes to all values  $r \leq n/2$  with a loss of a multiplicative factor of 2. This theorem was proven by Midrijānis [Mid05], and it gives a simple matrix with rigidity  $\mathcal{R}_{M_n}^{\mathbb{F}}(r) \geq \frac{n^2}{8r}$ .

We will see later that there exist matrices with much higher rigidity  $\tilde{\Omega}((n - r)^2)$ . Embarrassingly, the best known lower bound for an *explicit* matrix improves on the  $\frac{n^2}{8r}$  bound only by a logarithmic factor.

## 1.2 Circuit Complexity

A circuit corresponds to a simple straight line program where every instruction performs a binary operation on two operands, each of which is either an input or the result of a previous instruction. The structure of this program is extremely simple: no loops, no conditional statements. Still, we know no functions in P (or even NP, or even  $\text{E}^{\text{NP}}$ ) that requires even  $3.1n$  binary instructions (“size”) to compute on inputs of length  $n$ . This is in sharp contrast with the fact that it is easy to *non-constructively* find such functions: simple counting arguments show a random function on  $n$  variables has circuit size  $\Omega(2^n/n)$  with probability  $1 - o(1)$  [Sha49b, Lup59b].

For small-depth circuits we know several strong lower bounds. (Note that when working with circuits of constant depth, we do not pose bounds on the fan-ins of the gates.) Depth-2 circuits (after a simple normalization) are just CNFs or DNFs. It is easy to see that the parity function  $\oplus_n$  of  $n$  inputs requires CNFs and DNFs of size  $\Omega(2^n)$ . For depth- $d$  circuits, we know a lower bound of  $2^{\Omega(n^{1/(d-1)})}$  [Häs86, HJP93, PPZ97, Bop97, PPSZ05, MW17]. Thus, for depth  $d = o(\log n / \log \log n)$  we have non-trivial lower bounds even if the fan-ins of the gates are unbounded. For circuits with fan-in 2, we know functions which cannot be computed by circuits of depth  $1.99 \log n$  [Nec66]. Thus, a problem on the frontier is

**Problem 1.3.** *Prove a lower bound of  $10n$  against circuits of depth  $10 \log n$ .*

*More generally, a lower bound of  $\omega(n)$  against circuits of depth  $O(\log n)$ .*

Super-linear lower bounds are not known even for linear circuits, *i.e.*, circuits consisting of only gates computing linear combinations of their two inputs. Note that every linear function with one output has a circuit of size  $n - 1$  (and depth  $\log n$ ). For linear circuits, we consider *linear transformations*, multi-output

functions of the form  $f(x) = Ax$  where  $A \in \mathbb{F}^{n \times n}$ . For a random matrix  $A \in \{0, 1\}^{n \times n}$ , the size of the smallest linear circuit computing  $Ax$  is  $\Theta(n^2/\log n)$  [Lup56] with probability  $1 - o(1)$ , but for explicitly-constructed matrices the strongest known lower bound is  $3n - o(n)$  [Cha94b]. This leads us to another problem on the frontier:

**Problem 1.4.** *Prove a lower bound of  $\omega(n)$  against linear circuits of depth  $O(\log n)$ .*

Formally, Problem 1.3 and Problem 1.4 are incomparable, as in the linear case we study a weaker computational model (which makes it easier to prove lower bounds), but are limited to proving lower bounds for a smaller class of problem (which makes it harder to prove lower bounds).

### 1.3 Circuits and Rigidity

In this section, we will present a seminal result of Valiant [Val77] showing that rigid matrices require log-depth circuits of super-linear size. We start with the definition of linear circuits.

**Definition 1.5** (Linear circuits). *Let  $\mathbb{F}$  be a field and  $n \in \mathbb{N}$ . A circuit  $C$  with  $n$  inputs and  $n$  outputs is a directed acyclic graph where  $n$  vertices have fan-in zero and are labeled by the inputs, all other vertices have fan-in two and are labeled with affine functions (over  $\mathbb{F}$ ) of their two inputs,  $n$  of these vertices are labeled as outputs. For every fixed input, the value at each node is computed by applying the corresponding functions. Such a circuit  $C$  naturally defines a linear map  $f: \mathbb{F}^n \rightarrow \mathbb{F}^n$ , and the corresponding matrix  $A \in \mathbb{F}^{n \times n}$  such that  $f(x) = Ax$ .*

*The depth  $d(C)$  of a circuit  $C$  is the length of the longest path in the circuit. The size  $s(C)$  of  $C$  is defined as the number of vertices in  $C$ .*

The following theorem shows a connection between lower bounds for linear circuits and matrix rigidity.

**Theorem 1.6.** *Let  $\mathbb{F}$  be a field, and  $A \in \mathbb{F}^{n \times n}$  be a family of matrices for  $n \in \mathbb{N}$ . If  $\mathcal{R}_A^{\mathbb{F}}(\varepsilon n) > n^{1+\delta}$  for constant  $\varepsilon, \delta > 0$ , then any  $O(\log n)$ -depth linear circuit computing  $x \rightarrow Ax$  must be of size  $\Omega(n \cdot \log \log n)$ .*

The proof of Theorem 1.6 repeatedly uses the following beautiful graph theoretic lemma due to Erdős, Graham, and Szemerédi [EGS76]: If  $G$  is a directed acyclic graph with  $s$  edges and of depth  $d$ , then there is a set of  $s/\log d$  edges whose removal decreases the depth of  $G$  by a factor of two. We will follow the proof of this lemma from [Vio09].

**Lemma 1.7** ([EGS76]). *Let  $G$  be an acyclic digraph with  $s$  edges and of depth  $d = 2^k$ . There exists a set of  $s/\log d$  edges in  $G$  such that after their removal, the longest path in  $G$  has length at most  $d/2$ .*

*Proof of Lemma 1.7.* For ease of exposition, we follow [Vio09] and define a *depth function*. Let  $G = (V, E)$  be an acyclic digraph. We say that  $D: V \rightarrow \{0, 1, \dots, d\}$  is a depth function for  $G$  if for any  $(a, b) \in E$ ,  $D(a) < D(b)$ . It is not difficult to see that  $G$  has depth at most  $d$  if and only if there exists a depth function  $D: V \rightarrow \{0, 1, \dots, d-1\}$  for  $G$ .

We start with  $G$  of depth at most  $d = 2^k$ , and its depth function  $D: V \rightarrow \{0, 1, \dots, 2^k\}$ . Now, consider the following partition of  $E$  using the depth function  $D$ . For each  $i \in [k]$ , define

$$E_i = \{(a, b) \in E : \text{the most significant bit where } D(a), D(b) \text{ differ is the } i^{\text{th}} \text{ bit}\}.$$

As  $\{E_i\}_{i \in [k]}$  is a partition of  $E$ , by the averaging argument, there exists  $i^* \in [k]$  such that

$$|E_{i^*}| \leq \frac{|E|}{k} \leq \frac{|E|}{\log d}.$$

Now, it suffices to show that the depth of  $G' = (V, E')$ , where  $E' = E \setminus E_{i^*}$ , is at most  $2^{k-1}$ . This can be shown by exhibiting a depth function  $D': V \rightarrow \{0, 1, \dots, 2^{k-1} - 1\}$  for  $G'$ . The following shows that we can take  $D'(v)$  to be  $D(v)$  without the  $i^{\text{th}}$  bit.

Consider an edge  $(a, b) \in E'$ . Since  $(a, b) \in E$ ,  $D(a) < D(b)$ . In particular, there exists  $i \in [k]$  such that the most significant bit where  $D(a)$  and  $D(b)$  differ is  $i$ . Since  $(a, b) \in E'$ , the edge  $(a, b)$  was not removed, so  $i \neq i^*$ . Therefore, after removing the bit  $i^*$ , this bit  $i$  is still the most significant bit where  $D'(a)$  and  $D'(b)$  differ. This implies that  $D'(a) < D'(b)$ , and that  $D': V \rightarrow \{0, 1, \dots, 2^{k-1} - 1\}$  is a depth function for  $G'$ .  $\square$

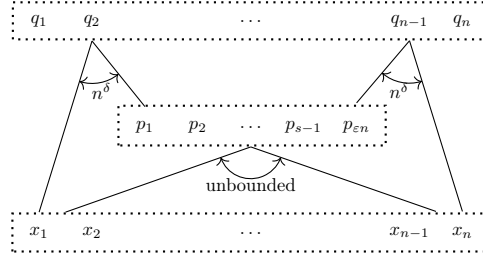


Figure 1.1: In order to compute the values of the outputs of the circuit  $C$ , first we precompute the values of  $\varepsilon n$  removed edges (or vertices  $V'$ ), and store them in variables  $p_i$ . Now each output  $q_j$  of the circuit  $C$  can be computed from  $n^\delta$  inputs and precomputed bits. In particular,  $A = BM + C$ , where  $C$  encodes the dependence of the outputs  $q_i$  on the inputs  $x_j$ ;  $B$  encodes the dependence of  $q_i$  on  $p_j$ ;  $M$  encodes the dependence of  $p_i$  on  $x_j$ . Since  $C$  is sparse, and  $BM$  is low rank, the matrix  $A$  is not rigid.

Now we finish the proof of [Theorem 1.6](#).

Lecture 2

*Proof of Theorem 1.6.* We will show that for every constant  $c_d \geq 2$ , every circuit of depth at most  $c_d \log n$  computing  $x \rightarrow Ax$  must be of size at least  $c_s n \log \log n$  for a constant  $c_s = \frac{\varepsilon}{\log c_d + \log 1/\delta}$ . Suppose, to the contrary, that there is a linear circuit  $C$  of size  $s = c_s n \log \log n$  and depth  $d = c_d \log n = 2^k$  that computes  $x \rightarrow Ax$ . Let  $G$  be the underlying acyclic digraph of  $C$ .

First, we apply [Lemma 1.7](#) to  $G$   $t$  times, and get a graph  $G'$  such that (i) only

$$s \cdot \left( \frac{1}{\log d} + \frac{1}{\log d - 1} + \cdots + \frac{1}{\log d - (t - 1)} \right) \leq \frac{st}{\log d - (t - 1)}$$

edges are removed from  $G$  and (ii) the longest path in  $G'$  is of length at most  $d' \leq d/2^t$ .

By setting  $t = \log c_d + \log 1/\delta$ , the longest path in  $G'$  has length  $\leq d/2^t = \delta \log n$ , and the number of removed edges is at most

$$\frac{st}{\log d - (t - 1)} = \frac{st}{\log d/2} \leq \frac{tc_s n \log \log n}{\log \log n} = \varepsilon n.$$

Now, let  $E$  be the set of removed edges and  $V'$  be the set of *tail* vertices of the edges from  $E$ . Since all paths in  $G'$  are no longer than  $d'$  and all in-degrees are at most 2, every output vertex in  $G'$  is now connected to at most  $2^{d'}$  input variables. Therefore, every output is a (linear) function of at most  $2^{d'}$  inputs and the functions computed at the removed edges (or the vertices  $V'$ ).

More specifically, let  $A_i$  be the  $i^{\text{th}}$  row of  $A$ , *i.e.*, the linear form computed by the  $i^{\text{th}}$  output vertex of  $G$ . Then  $A_i$  can be written as the following sum

$$A_i = \sum_{j \in [|V'|]} b_{ij} v_j + c_i$$

where  $v_j$  is the linear form computed by the  $j^{\text{th}}$  element in  $V'$  and  $c_i$  is the linear form computed by the  $i^{\text{th}}$  output vertex in  $G'$ . Note that since  $c_i$  only depends on at most  $2^{d'}$  input variables.

Therefore, the matrix  $A$  can be written as follows.

$$A = BM + C$$

where  $B \in \mathbb{F}^{n \times |V'|}$  consists of the coefficients  $b_{ij}$ , rows of  $M \in \mathbb{F}^{|V'| \times n}$  compute linear forms of vertices from  $V'$ , and  $C \in \mathbb{F}^{n \times n}$  is a row sparse matrix where the number of non-zero entries in each row is at most  $2^{d'} = n^\delta$ .

The above argument gives us that  $\tilde{\mathcal{R}}_A^{\mathbb{F}}(|V'|) = \tilde{\mathcal{R}}_A^{\mathbb{F}}(\varepsilon n) \leq n^\delta$ , which contradicts the assumption on the rigidity of  $A$ . □

## 1.4 Existence of Rigid Matrices

In this section, we will show that for any field  $\mathbb{F}$ , most of the  $n \times n$  matrices have the highest possible rigidity for any rank parameter  $r$ .

It turns out that for every matrix  $A$  and field  $\mathbb{F}$ , there is a simple upper bound  $\mathcal{R}_A^{\mathbb{F}}(r) \leq (n-r)^2$ . Valiant [Val77] showed that this upper bound is essentially tight for a random matrix. First, we give a proof of the upper bound.

**Theorem 1.8** (Simple upper bound). *For any field  $\mathbb{F}$ , matrix  $A \in \mathbb{F}^{n \times n}$ , and integer  $0 \leq r \leq n$ , we have that*

$$\mathcal{R}_A^{\mathbb{F}}(r) \leq (n-r)^2.$$

*Proof of Theorem 1.8.* If  $\text{rank}(A) \leq r$ , then  $\mathcal{R}_A^{\mathbb{F}}(r) = 0 \leq (n-r)^2$ . Thus, it suffices to focus on the case where there is an  $r \times r$  full-rank submatrix  $B \in \mathbb{F}^{r \times r}$  of  $A$ . Without loss of generality, assume that  $B$  is located in the top left corner of  $A$ :

$$A = \begin{pmatrix} B & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad (1.9)$$

where  $A_{12} \in \mathbb{F}^{r \times (n-r)}$ ,  $A_{21} \in \mathbb{F}^{(n-r) \times r}$ ,  $A_{22} \in \mathbb{F}^{(n-r) \times (n-r)}$ . In order to prove that  $\mathcal{R}_A^{\mathbb{F}}(r) \leq (n-r)^2$ , we will show that it is possible to change the entries in  $A_{22} \in \mathbb{F}^{(n-r) \times (n-r)}$  and reduce the rank of  $A$  to  $r$ . Since  $B$  has full rank, each row in  $A_{21}$  is a *unique linear combination* of the rows in  $B$ . Thus, we can change the entries in  $A_{22}$  according to these linear combinations so that each row in  $A$  is now a linear combination of the first  $r$  rows, *i.e.*, the rank of the modified matrix is at most  $r$ .<sup>2</sup>

Note that the above algorithm only modifies the entries of  $A_{22} \in \mathbb{F}^{(n-r) \times (n-r)}$ . Thus, at most  $(n-r)^2$  many entries in  $A$  are changed, and  $\mathcal{R}_A^{\mathbb{F}}(r) \leq (n-r)^2$ .  $\square$

We will now prove that almost all matrices have rigidity  $(n-r)^2$ .

**Theorem 1.10** (Valiant's lower bounds [Val77]). *For any field  $\mathbb{F}$ ,*

- *if  $\mathbb{F}$  is infinite, then for all  $0 \leq r \leq n$  there exists a matrix  $M \in \mathbb{F}^{n \times n}$  of rigidity*

$$\mathcal{R}_M^{\mathbb{F}}(r) = (n-r)^2;$$

- *if  $\mathbb{F}$  is finite, then for all  $0 \leq r \leq n - \Omega(\sqrt{n})$  there exists a matrix  $M \in \mathbb{F}^{n \times n}$  of rigidity*

$$\mathcal{R}_M^{\mathbb{F}}(r) = \Omega((n-r)^2 / \log n).$$

*Proof of Theorem 1.10.* Let  $M_{r,s} = \{A \in \mathbb{F}^{n \times n} : \mathcal{R}_A^{\mathbb{F}}(r) \leq s\}$  be the set of all matrices of  $r$ -rigidity at most  $s$ . We will show that the  $n^2$  elements of matrices from  $M_{r,s}$  lie in the union of images of a few rational maps from  $\mathbb{F}^{n^2+s-(n-r)^2}$  to  $\mathbb{F}^{n^2}$ . Intuitively, since for  $s \ll (n-r)^2$  these images cover only a negligible fraction of all matrices in  $\mathbb{F}^{n \times n}$ , we will have that “most” of the matrices are rigid.

For every matrix  $M \in M_{r,s}$ , there exists an  $s$ -sparse matrix  $S \in \mathbb{F}^{n \times n}$  and a low-rank matrix  $L \in \mathbb{F}^{n \times n}$ ,  $\text{rank}(L) = k \leq r$  such that  $M = S + L$ . After one of at most  $\binom{n}{k}^2$  permutations of rows and columns, we have the first  $k$  rows and columns of  $L$  linearly independent. The same permutations of rows and columns applied to  $M$ , give us a matrix of the form

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad (1.11)$$

where  $M_{11} \in \mathbb{F}^{k \times k}$ ,  $M_{12} \in \mathbb{F}^{k \times (n-k)}$ ,  $M_{21} \in \mathbb{F}^{(n-k) \times k}$ ,  $M_{22} \in \mathbb{F}^{(n-k) \times (n-k)}$ . Moreover, for at least one out of  $\binom{n^2}{s}$  choices of  $s$  entries of the matrix, we have that a change in those entries makes  $\text{rank}(M_{11}) = \text{rank}(M)$ . Similarly to Theorem 1.8, this implies that all entries of  $M_{22}$  are then rational maps of the entries in

<sup>2</sup>Formally, we set  $A_{22} = A_{21}B^{-1}A_{12}$ .

$M_{11}, M_{12}, M_{21}$ . That is, the  $n^2$  entries of any matrix  $M \in M_{r,s}$  lie in the union of at most  $\binom{n}{r}^2 \cdot \binom{n^2}{s}$  rational maps from  $\mathbb{F}^{s+n^2-(n-r)^2}$  to  $\mathbb{F}^{n^2}$ .

When  $\mathbb{F}$  is infinite and  $s < (n-r)^2$ , every matrix in  $M_{r,s}$  is in the union of finitely many images of rational functions from  $\mathbb{F}^{n^2-1}$  to  $\mathbb{F}^{n^2}$ . Since  $n^2$  rational functions of  $n^2-1$  variables are algebraically dependent (see, e.g., [For92]), a finite union of such images is the set of roots of a non-zero polynomial. This implies that some matrices from  $\mathbb{F}^{n \times n}$  do not belong to  $M_{r,s}$ .

When  $|\mathbb{F}| = q < \infty$  is finite, each  $M \in M_{r,s}$  is uniquely specified by one out of  $\binom{n}{r}^2$  permutations, one of  $\binom{n^2}{s}$  choices of  $s$  elements, values of those  $s$  elements, and values of the entries in  $M_{11}, M_{12}, M_{21}$ . Thus, the size of  $M_{r,s}$  is bounded from above by

$$\binom{n}{r}^2 \cdot \binom{n^2}{s} \cdot q^s \cdot q^{n^2-(n-r)^2} \leq 2^{2n+2s \log n} \cdot q^{n^2+s-(n-r)^2},$$

which is at most  $o(q^{n^2})$  for every  $s < (n-r)^2/\Omega(\log_q n)$  and  $r = n - \Omega(\sqrt{n})$ .  $\square$

Note that the proof of the  $\tilde{\Omega}((n-r)^2)$  lower bound in [Theorem 1.10](#) does not provide a description of a rigid matrix, it merely proves its existence. This brings us to a discussion on explicitness of matrix constructions.

Lecture 3

## 1.5 On explicitness

In this section we will see two constructions of very rigid matrices. The main drawback of these constructions is that we do not know a polynomial time algorithm outputting the entries of these matrices.

First we show that a matrix consisting of algebraically independent elements has maximal rigidity. (One simple way to construct  $n^2$  algebraically independent elements is given by Lindemann-Weierstrass Theorem.)

**Lemma 1.12.** *Let  $M \in \mathbb{R}^{n \times n}$  be a matrix where all  $n^2$  elements are algebraically independent over  $\mathbb{Q}$ . Then for every  $0 \leq r \leq n$ ,*

$$\mathcal{R}_M^{\mathbb{R}}(r) = (n-r)^2.$$

*Proof.* Let  $s = (n-r)^2 - 1$ . Assume, for the sake of contradiction, that  $\mathcal{R}_M^{\mathbb{R}}(r) \leq s$ . Then there exists an  $s$ -sparse matrix  $S$ , such that  $\text{rank}(M+S) \leq r$ . Similarly to [Theorem 1.10](#), the  $n^2$  entries of  $M$  are rational functions of the  $s$  non-zero entries of  $S$  and at most  $n^2 - (n-r)^2$  entries of  $M$ . Therefore, polynomials of at most  $s + n^2 - (n-r)^2 < n^2$  elements generate all  $n^2$  entries of  $M$ , which contradicts the assumption on algebraic independence of the elements of  $M$ . This implies that  $\mathcal{R}_M^{\mathbb{R}}(r) \geq (n-r)^2$ . On the other hand, [Theorem 1.8](#) gives us that  $\mathcal{R}_M^{\mathbb{R}}(r) \leq (n-r)^2$ .  $\square$

While the construction of [Lemma 1.12](#) has optimal rigidity, and each entry of such a matrix may have a very succinct mathematical description, there is no efficient algorithm outputting all digits of these entries. Thus, we will require that *explicit* constructions of matrices have polynomial-time algorithms outputting their entries.

Another non-explicit construction of rigid matrices is via an exponential-time algorithm. Suppose that we have a fixed finite field  $\mathbb{F}$  of size  $|\mathbb{F}| = q$ . Then there is a trivial algorithm which runs in time  $q^{O(n^2)}$  and outputs a rigid matrix. Let us fix an  $0 \leq r \leq n - \Omega(\sqrt{n})$ , and  $s = \Omega((n-r)^2/\log n)$ . In time  $q^{O(n^2)}$ , one can go over all pairs of matrices  $M, S \in \mathbb{F}^{n \times n}$ . For every such pair, the algorithm checks whether  $S$  is  $s$ -sparse and  $\text{rank}(M+S) \leq r$ . When the algorithm finds an  $M$  for which there is no  $S$  with the above conditions, it outputs  $M$  as a rigid matrix and halts. ([Theorem 1.10](#) guarantees existence of such a rigid matrix.)

When the field  $\mathbb{F}$  is infinite, an algorithm cannot enumerate all matrices. But even in this case it is possible to construct a rigid matrix in time  $2^{O(n^2)}$ . In order to prove this, we will first show that there exists a rigid matrix with all entries from  $\{0, 1\}$ . This will give us a way to enumerate all such matrices in time  $2^{O(n^2)}$ . Next, we will show that given such a matrix, one can check its rigidity in time  $2^{O(n^2)}$ , which will finish the proof.

**Theorem 1.13** ([PR94]). *For all large enough  $n$ , there exists a matrix  $M \in \{0, 1\}^{n \times n}$  such that*

$$\mathcal{R}_M^{\mathbb{R}}\left(\frac{n}{200}\right) \geq \frac{n^2}{100}.$$

*Proof.* Let  $r = \frac{n}{200}$ , and assume that all matrices  $M \in \{0, 1\}^{n \times n}$  have rigidity  $\mathcal{R}_M^{\mathbb{R}}(r) \leq s$ . We will show the lower bound of  $s \geq \frac{n^2}{100}$ . Each such matrix  $M$  can be written as

$$M = S + L_1 L_2, \tag{1.14}$$

where  $S \in \mathbb{R}^{n \times n}$  is  $s$ -sparse,  $L_1 \in \mathbb{R}^{n \times r}$  and  $L_2 \in \mathbb{R}^{r \times n}$ . There are  $\binom{n^2}{\leq s}$  ways to choose the set of non-zero entries in  $S$ , let us fix one such set  $\Gamma$ . From Equation 1.14, each entry of  $M$  is a degree-2 polynomial of the entries of  $L_1, L_2$  and  $\Gamma$ . In particular, there exist a set of  $n^2$  degree-2 polynomials  $\{f_{ij}^{\Gamma}\}_{i,j \in [n]}$  with variables being the entries of  $L_1, L_2, \Gamma$  such that  $M_{ij} = f_{ij}^{\Gamma}(L_1, L_2, \Gamma)$ .

For a set of  $t$ -variate polynomials  $F = \{f_{ij}\}_{i,j \in [n]}$ , we define its set of *zero-patterns* as the set of all sequences of zero-non-zero outputs of functions from  $F$ :

$$Z(F) = \{M \in \{0, 1\}^{n \times n} : \exists x \in \mathbb{R}^t \forall i, j \in [n], M_{i,j} = \mathbf{1}_{f_{ij}(x) \neq 0}\}.$$

We will use the following lemma which asserts that for a set  $F$  of low-degree polynomials,  $Z(F)$  is small.

**Lemma 1.15** ([RBG01]). *If  $F = \{f_{ij}\}_{i,j \in [n]}$  is a collection of  $t$ -variate polynomials of degree at most  $d$ , then*

$$|Z(F)| \leq \binom{t + dn^2}{t}.$$

*Proof of Lemma 1.15.* Let  $m = n^2$  be the number of polynomials, and let  $N = |Z(F)|$ . Let  $x_1, x_2, \dots, x_N \in \mathbb{R}^t$  be a set of points witnessing the  $N$  distinct zero-patterns of  $F$ . For an  $i \in [N]$ , let  $S_i \subseteq [m]$  be the set of (indices of) polynomials from  $F$  which are not zeros at the point  $x_i$ . For every  $i \in [N]$ , define the following polynomial

$$g_i = \prod_{k \in S_i} f_k.$$

Note that  $g_i(x_j) = 0$  if and only if there exists  $f_k \in S_i \setminus S_j$ . Therefore, we have  $g_i(x_j) = 0$  if and only if  $S_i \not\subseteq S_j$ .

Now we prove that all  $\{g_i\}_{i \in [N]}$  are linearly independent. Suppose, to the contrary, that there exist  $a_1, a_2, \dots, a_N \in \mathbb{R}$  such that  $\sum_{i \in [N]} a_i g_i = 0$ , and at least one  $a_i \neq 0$ . Let

$$i^* = \arg \min_{i \in [N], a_i \neq 0} |S_i|.$$

We have that  $a_{i^*} g_{i^*}(x_{i^*}) \neq 0$  and  $\sum_{i \in [N]} a_i g_i(x_{i^*}) = 0$ . Due to the minimality of  $S_{i^*}$ , for every  $a_i \neq 0$ ,  $S_i \not\subseteq S_{i^*}$ . This implies that  $a_i g_i(x_{i^*}) = 0$  for all  $i \neq i^*$ , and, thus,  $\sum_{i \in [N]} a_i g_i(x_{i^*}) \neq 0$ , which leads to a contradiction.

Finally, since the degree of each  $g_i$  is at most  $dm$ , and all  $g_i$  are linearly independent,  $N$  is bounded from above by the dimension of the space spanned by  $t$ -variate polynomials of degree at most  $dm$ . Thus,  $N \leq \binom{t+dm}{t}$ .  $\square$

Recall that from Equation 1.14, all matrices  $M \in \{0, 1\}^{n \times n}$  can be described by  $\{f_{ij}^{\Gamma}\}_{i,j \in [n]}$  for some  $\Gamma \in \binom{[n] \times [n]}{s}$ , where each polynomial has degree at most 2 and depends on  $2rn + s$  variables. Now, from Lemma 1.15 with  $t = 2rn + s$  and  $d = 2$ , we have that

$$\binom{2rn + s + 2n^2}{2rn + s} \cdot \binom{n^2}{\leq s} \geq 2^{n^2}. \tag{1.16}$$

Assume, for the sake of contradiction, that for some  $r \leq \frac{n}{200}$  we have  $s < \frac{n^2}{100}$ . We have that  $2rn + s \leq \frac{n^2}{50}$ . Now, the left-hand side of Equation 1.16 can be bounded from above as follows:

$$\binom{2rn + s + 2n^2}{2rn + s} \cdot \binom{n^2}{\leq s} \leq \binom{\frac{101n^2}{50}}{\frac{n^2}{50}} \cdot \binom{n^2}{\leq \frac{n^2}{100}} \leq (101e)^{\frac{n^2}{50}} \cdot (100e)^{\frac{n^2}{100}} \leq 2^{\frac{n^2}{2}},$$

which contradicts Equation 1.16. Thus, we conclude that for any  $r \leq \frac{n}{200}$ , there exists a matrix  $M \in \{0, 1\}^{n \times n}$  such that  $\mathcal{R}_M^{\mathbb{R}}(r) \geq \frac{n^2}{100}$ .  $\square$

Now we show that one can check whether a given matrix  $M \in \{0, 1\}^{n \times n}$  is rigid in time  $2^{O(n^2)}$ .<sup>3</sup>

**Theorem 1.17.** *Let  $M \in \{0, 1\}^{n \times n}$ , and  $r$  and  $s$  be non-negative integers. Then one can decide whether  $\mathcal{R}_M^{\mathbb{R}}(r) > s$  in time  $2^{O(n^2)}$ .*

*Proof.* Note that  $\mathcal{R}_M^{\mathbb{R}}(r) \leq s$  if and only if  $M = S + L_1 L_2$  for  $s$ -sparse  $S$  and  $L_1 \in \mathbb{R}^{n \times r}$ ,  $L_2 \in \mathbb{R}^{r \times n}$ . For any choice of non-zero entries of  $S$ , we have that the entries of  $M$  are degree-2 polynomials of  $t = 2nr + s$  variables with  $\{0, 1\}$ -coefficients. It is known that deciding whether such a system of polynomial equations has a real solution can be solved in time  $2^{O(n^2)}$  (see, e.g., Proposition 13.19 in [BPR07]). Since there are  $\binom{n^2}{s} \leq 2^{n^2}$  choices of  $s$  non-zero entries, we have that the total running time of the algorithm is  $2^{O(n^2)}$ .  $\square$

This way we have a set of  $2^{n^2}$  matrices such that at least one of them is rigid, and rigidity of each matrix can be checked in time  $2^{O(n^2)}$ . This gives us a  $2^{O(n^2)}$ -time algorithm for constructing a rigid matrix over the reals.

Although the above algorithms construct matrices of high rigidity, their running time is  $2^{\Omega(n^2)}$ . We define *explicit* constructions of matrices as matrices that have algorithms outputting all their entries in *polynomial time*.

## 1.6 Summary

In Theorem 1.8 we showed that for every field  $\mathbb{F}$ , matrix  $M \in \mathbb{F}^{n \times n}$ , and integer  $0 \leq r \leq n$ ,  $\mathcal{R}_M^{\mathbb{F}}(r) \leq (n-r)^2$ . Below we summarize the non-explicit *lower bounds on rigidity* presented in this chapter.

rigidity	field	running time	reference
$\frac{(n-r)^2}{\log n}$	any finite field	existence	Theorem 1.10
$(n-r)^2$	any infinite field	existence	Theorem 1.10
$(n-r)^2$	$\mathbb{R}$	algebraically independent entries	Lemma 1.12
$\frac{(n-r)^2}{\log n}$	any fixed finite field	$2^{O(n^2)}$	section 1.5
$(n-r)^2$	$\mathbb{R}$	$2^{O(n^2)}$	Theorem 1.13, Theorem 1.17

Table 1.1: Summary of non-explicit lower bounds.

<sup>3</sup>For more efficient algorithms for the case of low rigidity parameters see [FLM<sup>+</sup>18]. A **PSPACE**-algorithm for this problem follows immediately from the fact that existential theory of the reals lies in **PSPACE** [Can88].



## 1.7 Notes

Rigidity was introduced as a means to study circuit complexity by Valiant [Val77] and Grigoriev [Gri76]. An excellent presentation of the known lower bounds on rigidity over large fields can be found in the book of Lokam [Lok09]. The books of Jukna [Juk12] and Jukna and Sergeev [JS13] include many applications of matrix rigidity to circuit complexity. Earlier surveys on rigidity are due to Codenotti [Cod00] and Cheraghchi [Che05]. The tight upper bound of Theorem 1.8, and the non-constructive lower bounds of Theorem 1.10 and Theorem 1.13 were proven by Valiant [Val77], and Pudlák and Rödl [PR94]. The proof of Theorem 1.2 was first given by Midrijānis [Mid05].

# Chapter 2

## Explicit Constructions

In this chapter we give three proofs [Fri93, PR94, SSS97] of the best known explicit lower bound of  $\mathcal{R}(r) \geq \Omega\left(\frac{n^2}{r} \cdot \log \frac{n}{r}\right)$  on matrix rigidity. All the three proofs work for (almost) any generator matrix of a good linear code. Since there are explicit linear codes over all finite fields, the presented proofs work for all finite fields. We will see that the last construction (due to Shokrollahi, Spielman and Stemann) easily generalizes to infinite fields. Lecture 4

A linear code over a field  $\mathbb{F}$  is a linear subspace  $C \subseteq \mathbb{F}^n$  of dimension  $k$ . The *distance* of the code is the minimum Hamming distance between two vectors in  $C$  or, equivalently, the minimum Hamming weight of a non-zero vector in  $C$ .

**Definition 2.1** (Linear code). *Let  $\mathbb{F}$  be a field, and  $n, k, d$  be positive integers such that  $d, k < n$ . A subspace  $C \subseteq \mathbb{F}^n$  is a linear code of dimension  $k$  with minimum distance  $d$  if*

1.  $\dim(C) = k$ ;
2. for every  $x \in C \setminus \{0\}$ ,  $\|x\|_0 \geq d$ .

A linear code  $C \subseteq \mathbb{F}^n$  of dimension  $k$  can be specified by a *generator matrix*  $G \in \mathbb{F}^{n \times k}$  such that  $C$  is the column space of  $G$ . Since we focus on asymptotic behavior of codes, by a code we will mean an infinite sequence  $C = \{C_i : i \in \mathbb{N}\}$  where  $C_i \subseteq \mathbb{F}^n$ . For our purposes, it will suffice to say that a code is *explicit* if there is a polynomial time algorithm that for every  $n$ , outputs a generator matrix of  $C_n$  in time  $\text{poly}(n)$ . We will say that a code  $C$  is *good* if for the codes of this sequence we have that  $k = \Theta(n)$  and  $d = \Theta(n)$ .

There are exist explicit good linear error correcting codes over all finite fields (see, e.g., Justesen and Goppa codes in [MS77, LG88, vL12]).

**Proposition 2.2.** *For any finite field  $\mathbb{F}$ , there exists an explicit family of linear error correcting codes over  $\mathbb{F}$  of dimension  $k = n/4$  and minimum distance  $d = \delta n$  for a constant  $\delta > 0$ .*

The main result of this chapter is the following.

**Theorem 2.3.** *Let  $F$  be a fixed finite field, and  $G \in \mathbb{F}^{n \times k}$  be a generator matrix of a linear code of dimension  $k = \Theta(n)$  and distance  $d = \Theta(n)$ , then for every  $\Omega(\log n) < r < O(n)$ ,*

$$\mathcal{R}_G^{\mathbb{F}}(r) \geq \Omega\left(\frac{n^2}{r} \cdot \log \frac{n}{r}\right).$$

### 2.1 The lower bound of Friedman

Let us fix a generator matrix  $G \in \mathbb{F}^{n \times k}$  of a good linear code with distance and dimension  $d, k = \Theta(n)$ . We will prove the lower bound of Friedman in two steps. First, in [Theorem 2.4](#) we will show that  $G$  has high “column rigidity”. That is, in order to drop the rank of  $G$  to  $r$ , one has to modify at least  $\Omega\left(\frac{n}{r} \log_q \frac{k}{r}\right)$  entries in *some* column of  $G$ . Second, in [Theorem 2.5](#) we will use a simple averaging argument to reduce column rigidity to rigidity.

**Theorem 2.4** ([Fri93]). *Let  $\mathbb{F} = \mathbb{F}_q$  be a finite field of size  $q$ . Let  $G \in \mathbb{F}^{n \times k}$  be a generator matrix of a code of dimension  $k$  and distance  $\delta n$  for a constant  $0 < \delta < 1$ . For any  $\log_q k \leq r \leq \frac{k}{4}$ , if every column of  $B \in \mathbb{F}^{n \times k}$  contains at most  $\frac{\delta n}{4r} \log_q \frac{k}{r}$  non-zero entries, then*

$$\text{rank}(G + B) > r .$$

*Proof of Theorem 2.4.* Assume, for the sake of contradiction, that there exists  $B \in \mathbb{F}_q^{n \times k}$  such that  $\text{rank}(G + B) \leq r$  and each column of  $B$  has at most  $\frac{\delta n}{4r} \log_q \frac{k}{r}$  nonzero entries.

The proof employs two ideas. First, using a packing argument, we will show that the *kernel* of  $G + B$  must contain a sparse vector  $x \in \mathbb{F}^k$ . Second, as  $Gx$  is a codeword of  $C$  and  $Gx + Bx = \mathbf{0}$ ,  $Bx$  is also a codeword of  $C$  and, thus,  $\|Bx\|_0$  must be large due to the minimum distance property of  $C$ . This leads to a contradiction as  $x$  and the columns of  $B$  are sparse.

Let us draw a Hamming ball of radius  $d/2$  around each point in the kernel of  $G + B$ . Since we assume that  $\text{rank}(\ker(G + B)) = k - \text{rank}(G + B) \geq k - r$ , if

$$q^{k-r} \cdot |\text{Hamming ball of radius } d/2 \text{ in } \mathbb{F}_q^k| > q^k ,$$

then there must be two distinct points in the null space of  $G + B$  such that their Hamming balls intersect. This gives us a non-zero vector  $x$  in the kernel of  $G + B$  of sparsity at most  $d$ . The following shows that it suffice to pick  $d$  as an even number between  $\frac{2r}{\log_q \frac{k}{r}} \leq d \leq \frac{2r}{\log_q \frac{k}{r}} + 2$ .

$$\begin{aligned} |\text{Hamming ball of radius } d/2 \text{ in } \mathbb{F}_q^k| &\geq \binom{k}{d/2} \cdot (q-1)^{d/2} \\ &\geq q^{\frac{d}{2} \cdot \log_q \lfloor \frac{2k}{d} (q-1) \rfloor} > q^r . \end{aligned}$$

Next, since  $Gx$  is a *non-zero* codeword of  $C$  and  $Gx + Bx = \mathbf{0}$ , we know that  $Bx$  is a non-zero codeword of  $C$  and, thus,  $\|Bx\|_0 \geq \delta n$ . On the other hand,  $x$  has only  $d$  non-zero coordinates, and each column of  $B$  has at most  $\frac{\delta n}{4r} \log_q \frac{k}{r}$  non-zero entries. We have that

$$\|Bx\|_0 \leq d \cdot \frac{\delta n}{4r} \log_q \frac{k}{r} \leq \left( \frac{2r}{\log_q \frac{k}{r}} + 2 \right) \frac{\delta n}{4r} \log_q \frac{k}{r} < \delta n ,$$

which contradicts the distance property of  $C$ . □

**Theorem 2.5.** *Let  $\mathbb{F} = \mathbb{F}_q$  be a finite field of size  $q$ . Let  $G \in \mathbb{F}^{n \times k}$  be a generator matrix of a code of dimension  $k$  and distance  $\delta n$  for a constant  $0 < \delta < 1$ . Then for any  $\frac{\log_q k}{2} \leq r \leq \frac{k}{8}$ ,*

$$\mathcal{R}_G^{\mathbb{F}}(r) \geq \frac{\delta k n \log_q \frac{k}{2r}}{8r} .$$

*Proof.* Assume, for the sake of contradiction, that there exists  $S \in \mathbb{F}_q^{n \times k}$  such that  $\text{rank}(G + S) \leq r$  and  $\|S\|_0 \leq \frac{\delta k n \log_q \frac{k}{2r}}{8r}$ . Let  $J \subset [k]$  be the indices of the  $\frac{k}{2}$  sparsest columns of  $S$ , and let  $S_J$  be the sub-matrix of  $S$  restricted to the columns in  $J$ . By Markov's inequality, each column of  $S_J$  has at most

$$\left( \frac{\delta k n \log_q \frac{k}{2r}}{8r} \right) / \binom{k}{2} = \frac{\delta n}{4r} \log_q \frac{k}{2r}$$

many non-zero entries. Now Theorem 2.4 applied to  $G_J$  and  $S_J$  implies that a column in  $S_J$  must contain more than  $\frac{\delta n}{4r} \log_q \frac{k}{2r}$  non-zero entries, which leads to a contradiction. □

Now, using good codes from Proposition 2.2, we get a lower bound of  $\mathcal{R}_G^{\mathbb{F}}(r) \geq \Omega\left(\frac{n^2}{r} \cdot \log \frac{n}{r}\right)$  for any  $\Omega(\log_q n) \leq r \leq \frac{n}{32}$ .

## 2.2 The lower bound of Pudlák and Rödl

Lecture 5

**Theorem 2.6** ([PR94]). *Let  $\mathbb{F} = \mathbb{F}_q$  be a finite field of size  $q$ . Let  $G \in \mathbb{F}^{n \times k}$  be a generator matrix of a code of dimension  $k$  and distance  $\delta n$  for a constant  $0 < \delta < 1$ . Then for any  $1 \leq r \leq \frac{k}{q^2}$ ,*

$$\mathcal{R}_G^{\mathbb{F}}(r) \geq \frac{\delta k n \log_q \frac{k}{r}}{8r}.$$

*Proof of Theorem 2.6.* Assume that  $G = L + S$ , where  $\text{rank}(L) \leq r$  and  $\|S\|_0 \leq \frac{\delta n k}{4\ell}$  for an even integer  $\ell$  to be chosen later. Then, by Markov's inequality, there exist  $\frac{k}{2}$  columns of  $S$  each having at most  $\frac{\delta n}{2\ell}$  non-zero entries. Let  $G', L', S' \in \mathbb{F}^{n \times k/2}$  be the matrices  $G, L$  and  $S$  restricted to these columns. In particular, we have that  $\text{rank}(L') \leq r$ .

The proof is based on the following beautiful idea. By the distance property of the code, each non-zero linear combination of the columns of  $G'$  has weight at least  $\delta n$ . Now, since the columns of  $L' = G' - S'$  differ from the columns of  $G'$  only in a few positions, any *short* linear combination of the columns of  $L'$  is still non-zero. This guarantees that  $L'$  must generate a quite large space, which contradicts the assumption about the low rank of  $L'$ .

Observe that for any  $x \in \mathbb{F}_q^{k/2} \setminus \{0^{k/2}\}$  such that  $\|x\|_0 \leq \ell$ , we have

$$\|L'x\|_0 = \|(G' - S')x\|_0 \geq \|G'x\|_0 - \|S'x\|_0 \geq \delta n - \frac{\delta n}{2\ell} \cdot \ell = \frac{\delta n}{2} > 0,$$

where  $\|G'x\|_0 \geq \delta n$  follows from the fact that  $G'x$  is a non-zero codeword, and the other term follows from the column-sparsity of  $S'$ . This observation implies that for any distinct  $y_1, y_2 \in \mathbb{F}_q^{k/2}$  with  $\|y_1\|_0, \|y_2\|_0 \leq \frac{\ell}{2}$ ,  $L'y_1 \neq L'y_2$ .

Note that the above gives us a lower bound on the number of vectors in the column span of  $L'$ . Thus,

$$\text{rank}(L') \geq \log_q \binom{k/2}{\ell/2}.$$

Picking  $\frac{2r}{\log_q \frac{k}{r}} \leq \ell \leq \frac{2r}{\log_q \frac{k}{r}} + 2$  as an even integer, we have

$$\text{rank}(L') \geq \log_q \binom{k/2}{\ell/2} > \frac{\ell}{2} \log_q \frac{k}{\ell} \geq r,$$

which leads to a contradiction. Thus, we conclude that  $\mathcal{R}_G^{\mathbb{F}_q}(r) \geq \frac{\delta n k}{4\ell} \geq \frac{\delta n k}{8r} \log \frac{k}{r}$ .  $\square$

## 2.3 The lower bound of Shokrollahi, Spielman and Stemann

In this section we show that a few changes in a matrix always leave some large submatrix unchanged. Namely, if one makes only  $O(\frac{n^2}{r} \cdot \log \frac{n}{r})$  changes in an  $n \times n$  matrix, then there must be an untouched  $r \times r$  submatrix in it. In particular, if we start with a matrix whose every  $r \times r$  submatrix has high (or even full) rank, then the matrix remains high-rank even after  $O(\frac{n^2}{r} \cdot \log \frac{n}{r})$  changes, which implies rigidity.

First, in [Lemma 2.8](#) we will prove that after  $O(\frac{n^2}{r} \cdot \log \frac{n}{r})$  changes in  $n \times n$  matrix there always remains an untouched submatrix. Then, in [Theorem 2.9](#), we will apply this lemma to prove simple lower bounds on rigidity of explicit matrices, and in [Theorem 2.11](#) to prove lower bounds on the rigidity of (normalized) generator matrices of error-correcting codes.

We will need the classical Kővári-Sós-Turán theorem from extremal graph theory.

**Theorem 2.7** (Zarankiewicz problem [[KST54](#), [Bol04](#)]). *Let  $n, s \in \mathbb{N}$  such that  $s \leq n$  and  $G$  be an  $n \times n$  bipartite graph. If  $G$  has no  $s \times s$  bi-clique, then the number of edges in  $G$  is at most*

$$(s-1)^{1/s}(n-s+1)n^{1-1/s} + (s-1)n.$$

*Proof of Theorem 2.7.* Let  $G$  be an  $n \times n$   $K_{s,s}$ -free bipartite graph. Let  $\text{Star}_s = \{(u, T) \subseteq [n] \times \binom{[n]}{s} : (u, v) \in E(G), \forall v \in T\}$  be the set of left  $s$ -stars in  $G$ . There are two ways to count  $\text{Star}_s$ : (i) let  $d_1, d_2, \dots, d_n$  be the degrees of left vertices in  $G$ , then  $|\text{Star}_s| = \sum_{i \in [n]} \binom{d_i}{s}$  and (ii) for each  $T \in \binom{[n]}{s}$ ,  $T$  forms at most  $s-1$  many left  $s$ -star due to the  $K_{s,s}$ -free property. Namely,

$$\sum_{i \in [n]} \binom{d_i}{s} \leq |\text{Star}_s| \leq (s-1) \binom{n}{s}.$$

By convexity, this implies

$$\sum_{i \in [n]} (d_i - s + 1)^s \leq (s-1)(n-s+1)^s.$$

By Hölder's inequality,

$$\sum_{i \in [n]} (d_i - s + 1) \leq \left( \sum_{i \in [n]} (d_i - s + 1)^s \right)^{1/s} \cdot n^{1-1/s} \leq (s-1)^{1/s} (n-s+1) n^{1-1/s}.$$

Finally, as  $|E(G)| = \sum_{i \in [n]} d_i$ , we have that

$$|E(G)| = \sum_{i \in [n]} d_i = \sum_{i \in [n]} (d_i - s + 1) + (s-1)n \leq (s-1)^{1/s} (n-s+1) n^{1-1/s} + (s-1)n.$$

□

**Lemma 2.8.** *Let  $n, r \in \mathbb{N}$  such that  $\log n \leq r \leq n$ , and  $A$  be an  $n \times n$  matrix. If fewer than  $\frac{n(n-r)}{2(r+1)} \log \frac{n}{r}$  entries of  $A$  are changed, then some  $(r+1) \times (r+1)$  submatrix of  $A$  remains untouched.*

*Proof of Lemma 2.8.* Consider an  $n \times n$  bipartite graph  $G$ , whose vertices in one part correspond to the rows of  $A$ , and vertices in the other part correspond to the columns of  $A$ . We connect the vertex  $i$  from the first part to the vertex  $j$  from the second part if and only if the entry  $A_{ij}$  remains unchanged. Thus, an  $(r+1) \times (r+1)$  unchanged submatrix corresponds to an  $(r+1) \times (r+1)$  bi-clique in the bipartite graph  $G$ .

We assume, towards a contradiction, that there is no  $(r+1) \times (r+1)$  untouched submatrix in  $A$ , i.e., no  $(r+1) \times (r+1)$  bi-clique in  $G$ . We apply Theorem 2.7 with  $s = r+1$ , and conclude that a graph without an  $(r+1) \times (r+1)$  bi-clique has at most

$$\begin{aligned} r^{1/(r+1)}(n-r)n^{1-1/(r+1)} + rn &= n^2 - n(n-r) \cdot \left[ 1 - \left( \frac{r}{n} \right)^{1/(r+1)} \right] \\ &< n^2 - n(n-r) \frac{\log \frac{n}{r}}{2(r+1)} \end{aligned}$$

edges, where the last inequality uses the approximation  $e^{-x} < 1 - \frac{x}{2}$  for  $x \in [0, 1]$  and holds whenever  $r \geq \log n$ .

Finally, we conclude that if fewer than  $\frac{n(n-r)}{2(r+1)} \log \frac{n}{r}$  entries in  $A$  are changed, i.e.,  $G$  has more than  $n^2 - \frac{n(n-r)}{2(r+1)} \log \frac{n}{r}$  edges, then some  $(r+1) \times (r+1)$  submatrix of  $A$  remains untouched. □

An immediate corollary of Lemma 2.8 is that if every  $(r+1) \times (r+1)$  submatrix of  $A$  is full-rank, then  $\mathcal{R}_A(r) \geq \frac{n^2}{4(r+1)} \log \frac{n}{r}$  for  $\log n \leq r \leq \frac{n}{2}$ . The following theorem applies this idea to *Cauchy matrices* over small (but non-constant size) fields.

**Theorem 2.9** ([SSS97], non-fixed field). *Let  $\mathbb{F}$  be a field containing at least  $2n$  distinct elements denoted by  $x_1, x_2, \dots, x_n$  and  $y_1, y_2, \dots, y_n$ . Let  $A \in \mathbb{F}^{n \times n}$  be a Cauchy matrix:  $A_{ij} = \frac{1}{(x_i - y_j)}$ . Then*

$$\mathcal{R}_A^{\mathbb{F}}(r) \geq \frac{n^2}{4(r+1)} \log \frac{n}{r}$$

for  $\log n \leq r \leq \frac{n}{2}$ .

*Proof of Theorem 2.9.* By the above discussion, it suffices to show that every  $(r+1) \times (r+1)$  submatrix of the Cauchy matrix  $A$  has full rank for  $\log n \leq r \leq \frac{n}{2}$ . As every such submatrix is also a Cauchy matrix, it suffices to show that every Cauchy matrix has full rank.

In Problem 3 of Homework 1, we will show that the determinant of the Cauchy matrix  $A$  is

$$\det(A) = \frac{\prod_{1 \leq i < j \leq n} (x_j - x_i)(y_i - y_j)}{\prod_{1 \leq i, j \leq n} (x_i - y_j)}. \quad (2.10)$$

Since all  $x_i$  and  $y_j$  are distinct,  $\det(A) \neq 0$ , which finishes the proof.  $\square$

Finally, we use the above idea to construct (moderately) rigid matrices over constant-size fields.

**Theorem 2.11** ([SSS97], fixed field). *Let  $\mathbb{F}$  be a field,  $n \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ , and  $C \subseteq \mathbb{F}^{2n}$  be an explicit linear code of dimension  $n$  with minimum distance  $(1 - \varepsilon)n$ . Then, there exists a matrix  $A \in \mathbb{F}^{n \times n}$  that can be efficiently constructed from any generator matrix of  $C$  such that*

$$\mathcal{R}_A^{\mathbb{F}}(r) \geq \frac{n^2}{8(r+1)} \log \frac{n}{(2r+1)}$$

for any  $\varepsilon n \leq r \leq \frac{n-2}{2}$ .

*Proof of Theorem 2.11.* Let  $G \in \mathbb{F}^{2n \times n}$  be a generator matrix of  $C$ . We run Gaussian elimination in polynomial time to write  $G$  in the *standard form*:

$$G' = \begin{pmatrix} I_n \\ A \end{pmatrix},$$

where  $I_n$  is the  $n \times n$  identity matrix and  $A \in \mathbb{F}^{n \times n}$ .

**Claim 2.12.** *Let  $s = r + 1$ . Then, every  $2s \times 2s$  submatrix of  $A$  has rank at least  $s$ .*

*Proof of Claim 2.12.* Assume that there is a  $2s \times 2s$  submatrix of  $A$  of rank less than  $s$ . Without loss of generality, let it be the submatrix  $A' \in \mathbb{F}^{2s \times 2s}$  in the top left corner of  $A$ . As  $\text{rank}(A') < s$ , there exists a linear combination of  $s$  columns of  $A'$  which equals 0. Since any linear combination of the columns of  $G'$  is a codeword of  $C$ , we have a codeword  $x \in \mathbb{F}^{2n}$  whose first  $n$  coordinates have  $s$  non-zeros, and the last  $n$  coordinates have at least  $2s$  zeros. This gives us a codeword  $x \in C$  of sparsity

$$\|x\|_0 \leq s + (n - 2s) = n - s = n - r - 1 < (1 - \varepsilon)n,$$

which contradicts the assumption on the distance of the code  $C$ .  $\square$

Finally, Lemma 2.8 implies that in order to drop the rank of  $A$  to  $r$ , one needs to make at least

$$\frac{n(n - 2r - 2)}{4(r + 1)} \log \frac{n}{2r + 1} \geq \frac{n^2}{8(r + 1)} \log \frac{n}{2r + 1}$$

changes for any  $\varepsilon n \leq r \leq \frac{n-2}{4}$ .  $\square$

There are explicit constructions of algebraic-geometric codes [VT91, MS77] of dimension  $n$  in  $\mathbb{F}_q^{2n}$  with minimum distance  $(1 - \varepsilon)n$  for  $\varepsilon = \frac{2}{\sqrt{q}-1}$  for every prime square  $q$ . In particular, for every prime square  $q > 25$ , Theorem 2.11 applied to the algebraic-geometric codes gives rigidity lower bounds over  $\mathbb{F}_q$  for some range of  $r = O(n)$ .

## 2.4 Rigidity of the Walsh-Hadamard Matrix

Lecture 6

**Definition 2.13.** For any  $N = 2^n$ , the Walsh-Hadamard matrix  $H_N \in \mathbb{C}^{N \times N}$  is defined as

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$H_N = H_2^{\otimes n},$$

where  $\otimes$  denotes the Kronecker product.

First, in Problem 4 of Homework 1 we prove that  $H_N$  is not rigid for large rank  $r \geq N/2$ . Then, in [Theorem 2.18](#) we will prove that for every rank  $r \leq N/2$ , the rigidity of  $H_N$  is at least  $\mathcal{R}_H^{\mathbb{C}}(r) \geq \frac{N^2}{4r}$ .

**Lemma 2.14** (Homework 1, Problem 4). Let  $H \in \mathbb{C}^{N \times N}$  be the Walsh-Hadamard matrix. Then

$$\mathcal{R}_H^{\mathbb{C}}(N/2) \leq N.$$

In Problem 5 of Homework 1 we will prove the following facts about matrix norms.

**Proposition 2.15.** Let  $M \in \mathbb{C}^{m \times n}$  be a matrix,  $k = \min(m, n)$ , and  $r = \text{rank}(M)$ . Let

$$\sigma_1(M) \geq \dots \geq \sigma_r(M) > \sigma_{r+1}(M) = \dots = \sigma_k(M) = 0$$

be the singular values of  $M$ . Then

- The Frobenius norm  $\|M\|_F = \left( \sum_{i=1}^m \sum_{j=1}^n |M_{i,j}|^2 \right)^{1/2} = \left( \sum_{i=1}^k \sigma_i^2(M) \right)^{1/2}$ .
- The spectral norm  $\|M\|_2 = \sigma_1(M)$ .
- If  $M'$  is a submatrix of  $M$ , then  $\sigma_i(M') \leq \sigma_i(M)$ . In particular,  $\|M'\|_2 \leq \|M\|_2$ .

We will also need the following lower bound on the rank of submatrices of the Walsh-Hadamard matrix due to Lokam [[Lok95](#)].

**Lemma 2.16** ([[Lok95](#)]). For any submatrix  $H' \in \mathbb{C}^{a \times b}$  of the Walsh-Hadamard matrix  $H \in \mathbb{C}^{N \times N}$ ,

$$\text{rank}(H') \geq ab/N.$$

*Proof.* By [Proposition 2.15](#),

$$\|H'\|_F^2 = \sum_{i=1}^k \sigma_i^2(H') \leq \text{rank}(H') \cdot \sigma_1^2(H') = \text{rank}(H') \cdot \|H'\|_2^2 \leq \text{rank}(H') \cdot \|H\|_2^2. \quad (2.17)$$

Since the absolute values of all entries in  $H'$  are 1,  $\|H'\|_F^2 = ab$ . From  $H \cdot H^T = N \cdot I_N$ , we have that  $\sigma_i(H) = \sqrt{N}$  for  $1 \leq i \leq N$ , and, thus,  $\|H\|_2 = \sqrt{N}$ .

Now, from [Equation 2.17](#),

$$\text{rank}(H') \geq \|H'\|_F^2 / \|H\|_2^2 = ab/N.$$

□

Now we are ready to present the best known lower bound on rigidity of the Walsh-Hadamard matrix due to de Wolf [[De 06](#)]. Later in the course, we will also prove an upper bound on rigidity of the Walsh-Hadamard matrix.

**Theorem 2.18** ([[De 06](#)]). Let  $H \in \mathbb{C}^{N \times N}$  be the Walsh-Hadamard matrix. For every  $r \leq N/2$ ,

$$\mathcal{R}_H^{\mathbb{C}}(r) \geq \frac{N^2}{4r}.$$

*Proof.* Let  $s = \mathcal{R}_H^{\mathbb{C}}(r)$ , and let  $S \in \mathbb{C}^{N \times N}$  be such that

$$\text{rank}(H + S) \leq r \text{ and } \|S\|_0 \leq s.$$

Then by an averaging argument, there exists a set of  $2r$  rows of  $S$  with at most  $2rs/n$  non-zero entries. If  $N \leq 2rs/N$ , then  $s \geq \frac{N^2}{2r}$  concludes the proof. If  $N > 2rs/N$ , then we consider a submatrix  $H' \in \mathbb{C}^{(2r) \times (N - 2rs/N)}$  of  $H$  where all entries of  $S$  are zeros. By [Lemma 2.16](#),

$$r \geq \text{rank}(H') \geq 2r(N - 2rs/N)/N,$$

which implies  $s \geq \frac{N^2}{4r}$ . □

In the table below we summarize the known rigidity lower bounds for explicit matrices.

rigidity	reference
$\frac{n^2}{r^4 \log^2 r}$	[PS88]
$\frac{n^2}{r^3 \log r}$	[Raz88]
$\frac{n^2}{r^2}$	[Alo94]
$\frac{n^2}{r^2}$	[Lok95]
$\frac{n^2}{256r}$	[KR98]
$\frac{n^2}{4r}$	[De 06]

Table 2.1: Lower bounds on the rigidity of the Walsh-Hadamard matrix.

## 2.5 Overview of Chapters 1 and 2

See the slides for Lectures 6 and 7 for an overview of the following tools (and their applications).

- Probabilistic Method
- Algebraic Independence
- Polynomial Method
- Zarankiewicz Problem
- Hölder's inequality
- Zero-patterns
- Hadamard Matrix
- Spectral Methods
- Error-Correcting Codes

Lecture 7

Lecture 8



# Chapter 3

## Semi-explicit constructions

Lecture 9

### 3.1 Constructions with independent entries

Shoup and Smolensky [SS91] used an algebraic dimension argument for proving super-linear lower bounds against linear circuits of sub-polynomial depth. Lokam [Lok00] extended their argument, and proved a lower bound on rigidity of Vandermonde matrices containing the powers of  $n$  algebraically independent entries. Later Lokam [Lok06] generalized these ideas for proving quadratic lower bounds on rigidity for linear rank of matrices with infinite precision in the entries. Kumar, Lokam, Patankar and Sarma [KLPS14] used algebraic geometry arguments to construct a matrix with optimal rigidity parameters for every  $r$  (at the expense of having infinite precision in the entries, too).

This week we will see some of these constructions. Namely, we will prove rigidity lower bounds for a Vandermonde matrix with algebraically independent entries [Lok00], and for a matrix consisting of square roots of distinct primes [Lok06]. See [LTV03, Lok09, KLPS14, GHIL16] for excellent overviews of the algebraic geometry approach to rigidity.

Recall from Lemma 1.12 that any matrix with  $n^2$  algebraically independent entries is rigid. We would like to reduce the number of algebraically independent entries needed for constructions of rigid matrices, as well as to weaken the assumption on their independence. In the next class we will show that  $n^2$  linearly independent (but not algebraically independent) numbers are also sufficient for high rigidity. Today we will show that even  $n$  algebraically independent entries are sufficient for (moderate) rigidity.

#### 3.1.1 Vandermonde matrix with algebraically independence entries

Today we will show that Vandermonde matrices  $V$  with powers of algebraically independent entries have rigidity  $\mathcal{R}_V^{\mathbb{F}}(r) \geq \Omega(n^2)$  for  $r \leq O(\sqrt{n})$ .

**Definition 3.1** (Vandermonde matrix). *A Vandermonde matrix  $V \in \mathbb{F}^{n \times n}$  is a matrix of the form  $V_{ij} = x_i^j$  for some  $x_1, x_2, \dots, x_n \in \mathbb{F}$ .*

We will prove that Vandermonde matrices  $V$  are rigid as follows. First, in Definition 3.3 we will introduce a measure of algebraic independence (over  $\mathbb{Q}$ ) of a set of numbers, called the Shoup-Smolensky dimension. Second, in Lemma 3.4 we will prove that the Shoup-Smolensky dimension of low-rank matrices is low. Finally, in Lemma 3.5 we will show that the Shoup-Smolensky dimension of  $V - S$  for any sparse matrix  $S$  is large. From this, in Theorem 3.2, we will conclude that  $V$  is rigid.

**Theorem 3.2** (Vandermonde matrix with algebraically independent entries is rigid). *Let  $x_1, \dots, x_n \in \mathbb{C}$  be algebraically independent over  $\mathbb{Q}$ , and let  $V \in \mathbb{C}^{n \times n}$  be the Vandermonde matrix  $V_{i,j} = x_i^j$ . For any  $1 \leq r \leq \frac{\sqrt{n}}{10}$ ,*

$$\mathcal{R}_V^{\mathbb{C}}(r) \geq n(n - 100 \cdot r^2)/2.$$

For example, when  $r \leq \varepsilon\sqrt{n}$  for a small enough  $\varepsilon > 0$ , we have  $\mathcal{R}_V^{\mathbb{C}}(r) \geq \Omega(n^2)$ . However, it is an interesting open problem to prove non-trivial rigidity lower bound for such Vandermonde matrices in the regime of  $r = \omega(\sqrt{n})$ .

**Definition 3.3** (Shoup-Smolensky dimension [SS91]). *For any  $t, n \in \mathbb{N}$  and  $A \in \mathbb{C}^{n \times n}$ , the  $t$ -Shoup-Smolensky dimension of  $A$ , denoted  $\dim_t^{SS}(A)$ , is the dimension of the vector space over  $\mathbb{Q}$  spanned by products of  $t$  distinct elements of  $A$ .*

To prove [Theorem 3.2](#), we will to show that rank- $r$  matrices have Shoup-Smolensky dimension  $\leq \binom{nr+t}{t}^2$ , while  $V$  even after  $s$  changes has Shoup-Smolensky dimension  $\geq (n - \frac{s}{t})^t$ .

**Lemma 3.4.** *For any  $t, n \in \mathbb{N}$ , and  $A \in \mathbb{C}^{n \times n}$  of rank  $r = \text{rank}(A)$ ,*

$$\dim_t^{SS}(A) \leq \binom{nr+t}{t}^2.$$

*Proof of Lemma 3.4.* Since  $A$  has rank  $r$ , there exists  $B, C^{\top} \in \mathbb{C}^{n \times r}$  such that  $A = BC$ . Then each entry of  $A$  can be written as a degree-2 polynomial where each monomial is of the form  $b_1 c_1$  for some  $b_1$  from  $B$  and  $c_1$  from  $C$ .

Thus, any  $t$ -wise product of entries from  $A$  is a degree- $2t$  polynomial where each monomial is of the form  $\prod_{i \in [t]} b_i c_i$  for some  $b_i$  from  $B$  and  $c_i$  from  $C$  for all  $i \in [t]$ . This implies that the vector space spanned by the products of  $t$  distinct elements of  $A$  is also spanned by these monomials. As the number of  $n$ -variate monomials of degree  $\leq d$  is  $\binom{n+d}{d}$ , we conclude that  $\dim_t^{SS}(A) \leq \binom{nr+t}{t}^2$ .  $\square$

**Lemma 3.5.** *Let  $x_1, \dots, x_n \in \mathbb{C}$  be algebraically independent over  $\mathbb{Q}$ , and let  $V \in \mathbb{C}^{n \times n}$  be the Vandermonde matrix  $V_{i,j} = x_i^j$ . For any  $1 \leq t \leq \frac{n}{2}$ ,  $1 \leq s < tn$ , and  $S \in \mathbb{C}^{n \times n}$  such that  $\|S\|_0 \leq s$ ,*

$$\dim_t^{SS}(V - S) \geq \left(n - \frac{s}{t}\right)^t.$$

*Proof of Lemma 3.5.* Let  $J \subset [n]$  be the indices of the  $t$  sparsest rows of  $S$ , and  $S_J, V_J$  to be the restrictions of  $S$  and  $V$  to the rows in  $J$ . By Markov's inequality, each row in  $S_J$  has at most  $\frac{s}{t}$  non-zero entries.

Now, for each row in  $V_J - S_J$ , there are at least  $(n - \frac{s}{t})$  unchanged entries of the form  $x_i^j$ . Note that if  $x_1, x_2, \dots, x_n$  are algebraically independent, then any collection of distinct monomials of these variables is linearly independent. As there are at least  $(n - \frac{s}{t})^t$  distinct monomials in the set of  $t$ -wise products of the entries from  $V_J - S_J$ , we conclude that  $\dim_t^{SS}(V) \geq (n - \frac{s}{t})^t$ .  $\square$

Finally, [Theorem 3.2](#) follows from [Lemma 3.4](#) and [Lemma 3.5](#).

*Proof of Theorem 3.2.* For the sake of contradiction, assume that  $V = L + S$  where  $\text{rank}(L) \leq r$  and  $\|S\|_0 \leq s$ . Let us set  $t = \frac{n}{2}$  and  $s = n(n - 100r^2)/2$ . By [Lemma 3.4](#), we know that

$$\dim_t^{SS}(L) \leq \binom{nr+t}{t}^2 \leq \left(\frac{e(nr + \frac{n}{2})}{\frac{n}{2}}\right)^{\frac{n}{2}} \leq (81r^2)^{\frac{n}{2}}.$$

By [Lemma 3.5](#),

$$\dim_t^{SS}(V - S) \geq \left(n - \frac{s}{t}\right)^t = (100r^2)^{\frac{n}{2}}.$$

Therefore,  $\dim_t^{SS}(V - S) > \dim_t^{SS}(L)$ , which contradicts the assumption that  $V = L + S$ .  $\square$

### 3.1.2 Matrix with square roots of distinct primes

Today we will show that a matrix  $A$  with square roots of  $n^2$  distinct primes has rigidity  $\mathcal{R}_A^{\mathbb{C}}(r) \geq \Omega(n^2)$  Lecture 10  
for every  $1 \leq r \leq \frac{n}{32}$ .

**Theorem 3.6.** *Let  $A \in \mathbb{C}^{n \times n}$  be a matrix with square roots of  $n^2$  distinct primes as its entries. For any  $1 \leq r \leq \frac{n}{32}$ ,*

$$\mathcal{R}_A^{\mathbb{C}}(r) \geq n(n - 16r).$$

Similarly to the proof of [Theorem 3.2](#), we will show that any matrix with square roots of distinct primes has large Shoup-Smolensky dimension even when some entries of the matrix are changed.

We will use the Besicovitch theorem [[Bes40](#)] about linear independence over  $\mathbb{Q}$ .

**Theorem 3.7** (Besicovitch [[Bes40](#)]). *Let  $a_1, a_2, \dots, a_m$  be  $m$  distinct square roots of square-free integers, then they are all linearly independent over  $\mathbb{Q}$ .*

**Lemma 3.8.** *Let  $A$  be an  $n \times n$  matrix with square roots of  $n^2$  distinct primes as its entries, and  $S \in \mathbb{C}^{n \times n}$  such that  $\|S\|_0 \leq s$ . For any  $1 \leq s, t \leq n^2$ ,*

$$\dim_t^{SS}(A - S) \geq \binom{n^2 - s}{t}.$$

*Proof of Lemma 3.8.* There are at least  $n^2 - s$  square roots of distinct primes in the matrix  $A - S$ . Therefore, there are at least  $\binom{n^2 - s}{t}$   $t$ -wise products of  $A - S$  resulting in distinct square roots of square-free integers. Then, by [Theorem 3.7](#), the  $t$ -Shoup-Smolensky dimension of  $A - S$  is at least  $\dim_t^{SS}(A - S) \geq \binom{n^2 - s}{t}$ .  $\square$

Now we finish the proof of [Theorem 3.6](#).

*Proof of Theorem 3.6.* Let us set  $t = nr$  and  $s \leq n(n - 16r)$ . Assume, for the sake of contradiction, that  $A = L + S$  where  $\text{rank}(L) \leq r$  and  $\|S\|_0 \leq s$ . From [Lemma 3.4](#),

$$\dim_t^{SS}(L) \leq \binom{nr + t}{t}^2 \leq \binom{2nr}{nr}^2 < 2^{2nr \cdot 2} = 16^{nr}.$$

On the other hand, from [Lemma 3.8](#),

$$\dim_t^{SS}(A - S) \geq \binom{n^2 - s}{t} \geq \left(\frac{16nr}{nr}\right)^{nr} = 16^{nr}.$$

Thus,  $\dim_t^{SS}(A - S) < \dim_t^{SS}(L)$ , which leads to a contradiction.  $\square$

### 3.1.3 Lower Bounds against Linear Circuits

While the rigidity lower bound from [Section 3.1.2](#) implies a super-linear circuit lower bound against linear circuits for a (not fully explicit) matrix  $A$  with square roots of distinct primes, the same technique can be used to directly prove stronger circuit lower bounds for the same matrix  $A$ . Shoup and Smolensky [[SS91](#)] proved that any linear circuit (of any depth) computing the linear transformation given by such a matrix  $A$  must have size at least  $\Omega(n^2 / \log n)$  (which is optimal as every linear function can be computed by a circuit of size  $O(n^2 / \log n)$  [[Lup56](#)]).

For simplicity, in this section for an  $n \times n$  matrix  $A$ , by its  $n^2$ -Shoup-Smolensky dimension  $\dim_{n^2}^{SS}(A)$  we will mean the dimension of the vector space over  $\mathbb{Q}$  spanned by products of *at most*  $n^2$  elements of  $A$  (rather than exactly  $n^2$  elements of  $A$ ). We will use the following technical lemma.

**Lemma 3.9.** *Let  $C$  be a linear circuit of size  $s$  computing  $x \rightarrow Bx$  for  $B \in \mathbb{C}^{n \times n}$ . Then*

$$\dim_{n^2}^{SS}(A) \leq (n^2 + 2s)^{2s}.$$

*Proof.* See the class video and slides for a full proof.  $\square$

Now we can prove an optimal circuit lower bound for matrices consisting of square roots of distinct primes.

**Theorem 3.10.** *Let  $A \in \mathbb{C}^{n \times n}$  be a matrix with square roots of  $n^2$  distinct primes as its entries. For any  $1 \leq r \leq \frac{n}{32}$ ,*

$$\mathcal{R}_A^C(r) \geq n(n - 16r).$$

*Proof.* By Besicovitch Theorem (Theorem 3.7), all  $2^{n^2}$  products of subsets of elements of  $A$  are linearly independent, therefore

$$\dim_{\mathbb{F}_2}^{SS}(A) \geq 2^{n^2}.$$

This bounds, together with the bound of Lemma 3.9, implies

$$s \geq \Omega(n^2 / \log n).$$

□

## 3.2 Project Topics

We discussed the following project ideas (see the slides and video for more details):

Lecture 11

- Provable Cryptography
- Static Data Structures
- Random Algebraic Method

## 3.3 Rigidity of Hankel and Toeplitz matrices

Lecture 12

**Definition 3.11** (Hankel/Toeplitz matrix).  $A \in \mathbb{F}^{n \times n}$  is a Hankel matrix if  $A_{i,j} = a_{i+j}$  for some  $a_2, a_3, \dots, a_{2n} \in \mathbb{F}$ .  $A \in \mathbb{F}^{n \times n}$  is a Toeplitz matrix if  $A_{i,j} = a_{i-j}$  for some  $a_{-(n-1)}, a_{-(n-2)}, \dots, a_{n-1} \in \mathbb{F}$ .

A random Hankel matrix  $A \in \mathbb{F}^{n \times n}$  is a Hankel matrix where  $a_2, a_3, \dots, a_{2n} \in \mathbb{F}$  are independent uniformly random elements of  $\mathbb{F}$ . In Theorem 3.13 we will prove a lower bound on the rigidity of a *random* Hankel matrix, and later in this course we will prove an upper bound on the rigidity of *all* Hankel matrices. These bounds naturally extend to the case of Toeplitz matrices. Note that only  $O(n)$  bits of random bits are needed to sample a random Hankel/Toeplitz matrix over  $\mathbb{F}_2$ . We will need the following generalization of Toeplitz matrices.

**Definition 3.12.**  $B \in \mathbb{F}^{n \times n}$  is a  $k$ -Hankel matrix for  $k \in [n]$  if  $B_{i,j} = b_{k(i-1)+j}$  for some  $b_1, b_2, \dots, b_{(n-1)k+n} \in \mathbb{F}$  for all  $i, j \in [n]$ .

For example, 1-Hankel matrix is a usual Hankel matrix from Definition 3.11. In particular, every row of a random 1-Hankel matrix has one new random element. A random  $n$ -Hankel matrix is a uniform random matrix with  $n^2$  random entries. In general, every row of a random  $k$ -Hankel matrix has  $k$  new random entries.

**Theorem 3.13** (Rigidity of a random Hankel matrix [GT16]). For any  $\sqrt{n} \leq r \leq \frac{n}{32}$ , a random Hankel matrix  $A \in \mathbb{F}^{n \times n}$  has

$$\mathcal{R}_A^{\mathbb{F}_2}(r) \geq \Omega\left(\frac{n^3}{r^2 \log n}\right)$$

with probability  $1 - o(1)$ .

The bound of Theorem 3.13 improves on the known *explicit* rigidity bounds (presented in chapter 2) for  $r = o\left(\frac{n}{\log n \log \log n}\right)$ . For example, for  $r = n^{1-\varepsilon}$  for a constant  $\varepsilon \in (0, \frac{1}{2})$ , Theorem 3.13 gives a lower bound of

$$\mathcal{R}_A^{\mathbb{F}_2}(n^{1-\varepsilon}) \geq \Omega\left(\frac{n^{1+2\varepsilon}}{\log n}\right),$$

while the bounds of chapter 2 give only  $\mathcal{R}_A^{\mathbb{F}_2}(n^{1-\varepsilon}) \geq \Omega\left(\frac{n^2}{r} \cdot \log \frac{n}{r}\right) \geq \Omega(n^{1+\varepsilon} \log n)$ .

We will prove Theorem 3.13 in two steps. Let  $n$  and  $k$  be integers such that  $k$  divides  $n$ . In the first step, we will show that any  $n \times n$  Hankel matrix can be *partitioned* into  $k^2 \frac{n}{k} \times \frac{n}{k}$  matrices, each of which is  $k$ -Hankel. In the second step we will prove that a random  $k$ -Hankel matrix is quite rigid with high probability. From these two facts, we will conclude that a random Hankel matrix has high rigidity (with high probability).

**Lemma 3.14** (Partition of a Hankel matrix). *Let  $n, k \in \mathbb{N}$  such that  $k$  divides  $n$ . Any Hankel matrix  $A \in \mathbb{F}^{n \times n}$  can be partitioned into  $k^2 \frac{n}{k} \times \frac{n}{k}$   $k$ -Hankel matrices.*

*Proof of Lemma 3.14.* Let  $m = \frac{n}{k}$ . For each  $i, j \in [k]$ , let us define the following sets of indices of rows and columns

$$\begin{aligned} I_i &= \{i, i+k, \dots, i+(m-1)k\}; \\ J_j &= \{(j-1)m+1, (j-1)m+2, \dots, jm\}. \end{aligned}$$

We partition the Hankel matrix  $A$  into  $k^2$  submatrices  $A^{i,j}$ ,  $i, j \in [k]$  as follows: the matrix  $A^{i,j}$  contains all entries of  $A$  at the intersections of the rows  $I_i$  and the columns  $J_j$ . It is easy to see that each  $A^{i,j} \in \mathbb{F}^{m \times m}$ , and that  $\{(I_i, J_j)\}_{i,j \in [k]}$  forms a partition for  $[n] \times [n]$ .

It remains to show that each  $A^{i,j}$  is  $k$ -Hankel. Consider the element of the matrix  $A^{i,j}$  located at the position  $(s, t)$ :

$$A_{s,t}^{i,j} = A_{i+(s-1)k, (j-1)m+t}.$$

Since  $A$  is a Hankel matrix, by Definition 3.11,  $A_{i+(s-1)k, (j-1)m+t} = a_{i+(j-1)m+(s-1)k+t}$ . Now, let  $b_\ell = a_{i+(j-1)m+\ell}$  for  $\ell \in [(m-1)k+m]$ . Then

$$A_{s,t}^{i,j} = A_{i+(s-1)k, (j-1)m+t} = a_{i+(j-1)m+(s-1)k+t} = b_{(s-1)k+t},$$

which satisfies Definition 3.12.  $\square$

**Lemma 3.15** (Rigidity of  $k$ -Hankel matrices). *For any  $16 \leq k \leq m$ , with probability  $1 - 2^{-\frac{km}{20}}$  a random  $m \times m$   $k$ -Hankel matrix  $A$  has rigidity*

$$\mathcal{R}_A^{\mathbb{F}_2}(m/2) \geq \frac{km}{400 \log m}.$$

Finally, we prove Theorem 3.13 using Lemma 3.14 and Lemma 3.15.

*Proof of Theorem 3.13.* We will prove this theorem for every  $\sqrt{n} \leq r \leq \frac{n}{32}$  such that  $2r$  divides  $n$ , the same result (with a larger constant factor hidden in the  $\Omega$ -notation) for other values of  $r$  will follow immediately. Let us set  $m = 2r$  and  $k = \frac{n}{m}$ .

From Lemma 3.14,  $A$  can be partitioned into  $m \times m$   $k$ -Hankel matrices  $\{A^{i,j}\}_{i,j \in [k]}$ . For the sake of contradiction, assume that  $A = S + L$  where  $\|S\|_0 \leq \frac{n^3}{1600r^2 \log n}$  and  $\text{rank}(L) \leq r$ . By averaging, there exist  $i, j \in [k]$  such that  $A^{i,j} = S^{i,j} + L^{i,j}$  where

$$\|S^{i,j}\|_0 \leq \frac{\|S\|_0}{k^2} \leq \frac{n}{400 \log n},$$

and  $\text{rank}(L^{i,j}) \leq \text{rank}(L) \leq r$ . In particular,  $A^{i,j}$  is not rigid:  $\mathcal{R}_A^{\mathbb{F}_2}(r) \leq \frac{n}{400 \log n}$ .

However, Lemma 3.15 and union bound imply that with probability  $1 - m^2 \cdot 2^{-\frac{km}{20}} = 1 - o(1)$ , for every  $i, j \in [k]$ ,  $\mathcal{R}_{A^{i,j}}^{\mathbb{F}_2}(\frac{m}{2}) \geq \frac{km}{400 \log m} = \frac{n}{400 \log m} > \frac{n}{400 \log n}$ . This contradicts the assumption that  $A = S + L$ , and, thus,  $A$  has rigidity  $\mathcal{R}_A^{\mathbb{F}_2}(r) \geq \frac{n^3}{1600r^2 \log n}$  with probability  $1 - o(1)$ .  $\square$

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