Matrix Rigidity

Rectangular PCPs

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Overview

- Recall that we want $M \in \mathbb{F}_2^{n\times n}$, $M \in \mathsf{E}^{\mathsf{NP}}$
$$\mathcal{R}_M^{\mathbb{F}_2}(\varepsilon n) \geq \Omega(n^{1+\delta})\,.$$
- We'll prove that there is $M \in \mathbb{F}_2^{n\times n}$, $M \in \mathsf{P}^{\mathsf{NP}}$!
$$\mathcal{R}_M^{\mathbb{F}_2}(2^{\log n/\log\log n}) \geq \Omega(n^2)\,.$$
- We'll use
  - Orthogonal Vectors
  - Non-deterministic Hierarchy Theorem
  - Rectangular PCPs

Rectangular PCPs

Constraint Satisfaction Problems (CSP)

Definition

A k-CSP is specified by a Boolean function $f\colon \{0,1\}^{\boxed{k}} \to \{0,1\}$. An instance of k-CSP is a formula with n Boolean variables and $m = \mathrm{poly}(n)$ constraints, where each constraint is f applied to a k-tuple of variables or their negations.

3-CSP 2-CSP

3-SAT

Ex. $k=3$ $f = \mathrm{OR}$

$(x_1 \vee \overline{x_2} \vee x_n)$

$(\overline{x_7} \vee x_{10} \wedge x_{11})$

Ex. $k=2$ $f = \mathrm{XOR}$

$(x_1 \oplus x_7)$

$(x_n \oplus \overline{x_5}) = (x_n \oplus x_5 \oplus 1)$

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Definition

In Max-kCSP, the goal is to find the maximum number of simultaneously satisfiable clauses.

SAT = can we satisfy all clauses of φ?

MAX-SAT – opt. version of SAT

Given φ, k, can we satisfy $\geq k$ clauses?

$$(x_1 \vee x_2) \wedge (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2})$$

is not satisfiable, SAT $\rightarrow$ no

MAX-SAT $\rightarrow$ 3

MAX-CSP

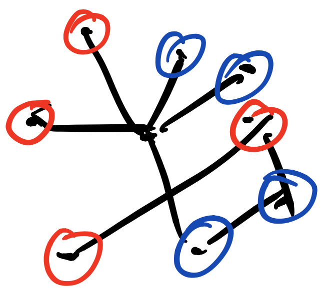
Ex. $k=2$ $\mathcal{F} = XOR$

$(x_1 \oplus x_7)$ $(x_n \oplus x_3)$ ...

MAX-2LIN / MAX-2XOR / MAX-CUT

MAX-CUT problem:

G



cut = # of red-blue edges

MAX-CUT - find a cut (red-blue coloring of vertices) that max # of cut edges (red-blue edges)

Previously: we showed very simple $\frac{1}{2}$-apx for MAX-CUT using randomness

Then we said, pairwise ind. (Toeplitz) derandomizes alg.

every vertex - $x_i \in \{0,1\}$

every edge $(x_1, x_2) \rightarrow (x_1 \oplus x_2)$

$x_i = 0$ if x_i is red

$x_i = 1$ if x_i is blue

sat iff edge is red-blue

G - instance of MAX-CUT

$\Downarrow$

write equivalent

MAX-2XOR fla. φ

Max # of sat clauses in φ

$=$

Max-CUT in G

PCP

- Motivation: Hardness of Approximation

PCP

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- Two views of PCPs:

PCP

- Motivation: Hardness of Approximation

- Two views of PCPs:
    - Probabilistically Checkable Proofs

PCP

- Motivation: Hardness of Approximation

- Two views of PCPs:
  - Probabilistically Checkable Proofs
  - Hardness of Approximation

Probabilistically Checkable Proofs

Recall: $L \in \mathbf{NP}$ if there exists deterministic poly-time V:

$$\begin{aligned} \underline{x \in L} &\implies \quad \underline{\exists \pi} \quad s.t. V^{\pi}(x) = 1 \\ \underline{x \notin L} &\implies \quad \underline{\forall \pi} \quad V^{\pi}\underline{(x)} = 0 \end{aligned}$$

$L = SAT \in NP$

$\varphi \in SAT \Rightarrow \underline{\exists x \in \{0,1\}^n} \quad V^{x}(\varphi) = 1$

$\varphi \notin SAT \Rightarrow \forall x \in \{0,1\}^n \quad V^{x}(\varphi) = 0$

Probabilistically Checkable Proofs

Recall: $L \in \mathbf{NP}$ if there exists deterministic poly-time V:

$$x \in L \implies \quad \exists \pi \quad s.t. V^{\pi}(x) = 1$$
$$x \notin L \implies \quad \forall \pi \quad V^{\pi}(x) = 0$$

In the class **PCP**, V is $\boxed{\text{randomized}}$ and has random access to π

PCP

randomness

queries

Let $r, q : \mathbb{N} \to \mathbb{N}$, then $\mathbf{PCP}(r, q)$ is the class of languages L s.t. there exists a poly-time probabilistic V:

- Efficient: For $x \in \{0,1\}^n$ and a proof π, V has $O(r(n))$ random bits, makes $O(q(n))$ queries to π
- Completeness: If $x \in L$, then V accepts with probability $\geq c = 1$. – perfect completeness
- Soundness: if $x \notin L$, then V accepts with probability $< s < c$.

$= \frac{1}{2}$

PCP for 3-SAT

φ with n vars
m clauses.

$\varphi \in SAT \iff \varphi$ is satisfiable.

Proof(Π) to be $x \in \{0,1\}^n$

$\varphi \in SAT \implies \exists \underline{x \in \{0,1\}^n}$
s.t. efficient V looks
at 1 random clause of φ.
$\implies$ accept φ.

$\varphi \notin SAT \implies \forall x \underline{\in \{0,1\}^n}$
V will detect
unsatisfied clause w.p. $\geq \frac{1}{m}$.

$\log m$ random bits, 3 queries

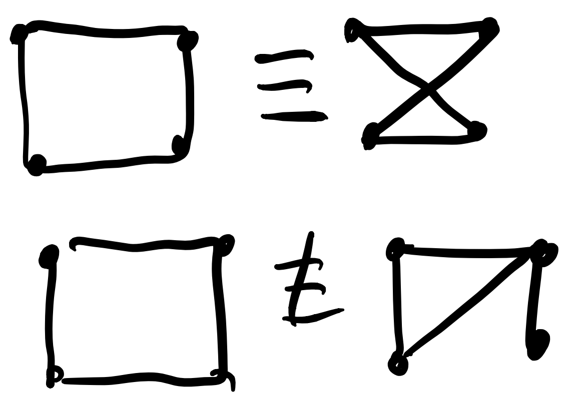
This was example of

$PCP(\log m, 3)$ $\underline{c = 1}$

$s = \underline{1 - \frac{1}{m}}$ for 3-SAT

$GNI(G_1, G_2)$

$G_1 \not\equiv G_2$



$GNI \in PCP(poly(n), 1)$

G_1, G_2 with n

Proof π: $\forall$ every graph on n vertices G.

0 if $G \equiv G_1$

1 if $G \equiv G_2$

0 if $G \not\equiv G_1$, $G \not\equiv G_2$

V has graphs G_1, G_2

picks random $b \in \{1, 2\}$: G_b.

Takes random perm $G_b \to H$

If $G_1 \not\equiv G_2$.

Say, $b=1$. $H \equiv \underline{\underline{G_1}}$

$\Pi(H) = 1$.

V accepts $G_1 \not\equiv G_2$

If $\underline{G_1 \equiv G_2}$

Say, $b=1$, $H \equiv G_1 \equiv G_2$

$\Pi(H) = 0$ or $\Pi(H) = 1$

V accepts w p $\frac{1}{2}$.

PCP THEOREM

Theorem (PCP Theorem [AS92, ALMSS92])

$$\underline{\underline{\mathrm{NP}}} = \mathrm{PCP}(\log n, 1)\,.$$

$O(\log n)$ randomness

$O(1)$ = const queries

$x \in L \Rightarrow$ accept always $\quad c = 1.$

s = any constant

$s = 0.9 \xRightarrow{\text{amplify}} s = 0.01$

$L = \{$ math statements of length n that have proofs of length poly(n) $\}$

$L \in NP$, for every statement there is a proof of length poly(n), Verifier can always check math proofs.

$L \in PCP(\log n, 1)$

Every math proof (of poly-length) can rewritten in some PCP form s.t. one can check correctness of the proof by looking at only 3 bits of the proof.

PCP Theorem

Theorem (PCP Theorem [AS92, ALMSS92])

$$\mathrm{NP} = \mathrm{PCP}(\underline{\log n}, \underline{1}) \,.$$

Theorem (Scaled-up PCP [BFLS91, AS92, ALMSS92])

$$\mathrm{NEXP} = \mathrm{PCP}(\underline{\mathrm{poly}(n)}, 1) \,.$$

<u>Ex.</u> $GNI \in PCP(\mathrm{poly}(n), 1)$

PCP: Hardness of Approximation

PCPT: NP = PCP(log n, 1)

Theorem (PCP Theorem)

There exists a constant $p < 1$ s.t. it's **NP**-*hard to p-approximate Max-3SAT.*

3SAT is NP-hard to solve exactly $\Rightarrow$ 3-SAT is NP-hard

cannot distinguish between 3-SAT φ : $\begin{cases} val(\varphi) = 1 \\ val(\varphi) \leq 0.75 + \varepsilon \end{cases}$

easier ~~than~~ than 3-SAT

Equivalence of Two Views

Example: $\mathbf{NP} = \mathbf{PCP}(\log n, 1)$ implies that k-CSP is $\mathbf{NP}$-hard to p-approximate for some constant p and k.

3-SAT $\in$ NP $\in$ PCP(log n, 1)

φ is a 3-CNF fla. $\exists\, \Pi \in \{0,1\}^n$ assignment $x_1 \ldots x_n$

$\forall\, R \in \{0,1\}^{10 \log n}$

V looks at 3 positions of Π

V decides whether accept/reject φ.

$\forall$ randomness $R \in \{0,1\}^{10 \log n}$

V looks at 3 positions

$x_1 \quad , x_7 \quad , x_{11}$

$(x_1 \wedge x_7 \vee \overline{x_{11}})$ 3-CSP - use whatever fla of 3 xs

$2^{10 \log n}$ different strings R.

" n^{10} clauses

$(x_1 \wedge x_7 \vee \overline{x_{11}})$

$(x_n \wedge x_{n/2} \vee \overline{x_2})$ .....

n^{10} clauses, 3-CSP

Verifier samples a random clause, and verifies whether it's satisfied by proof $x \in \{0,1\}^n$

PCP $c = 1$

$s \le \frac{1}{2}$

φ is SAT $\Rightarrow$ V accepts w.p. 1

φ is not SAT $\Rightarrow$ w.p. $\le \frac{1}{2}$

φ is SAT $\Rightarrow$ V accepts w.p. 1

$\Rightarrow$ every clause of 3-CSP can be sat

φ is not SAT $\Rightarrow$ V accept $\leq \frac{1}{2}$

$\Rightarrow$ $\leq \frac{1}{2}$ clauses can be satisfied.

Hardness of apx view of PCP

PCP starts with like 3SAT

hard $\begin{cases} m \text{ clauses} \\ \leq m-1 \text{ clauses} \end{cases} \Rightarrow \begin{cases} m \text{ clauses} \\ \leq 0.9m \text{ clauses} \end{cases}$

Equivalence of Two Views

Proof View	Approximation View

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PCP Verifier	CSP formula

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Equivalence of Two Views

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Length of proof	# of vars
# of queries	width of constraints
# of random bits	$\log$ of # of constraints

Smooth PCPs

$x_1, \ldots, x_n \in \{0,1\}^n$

no favorite variable

Definition

A PCP is smooth if V queries every variable with equal probability (queries every bit of the proof with equal probability).

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If your proof π is not correct, but differs from it in a few positions, chances are your proof will be accepted by a smooth verifier!

correct $x_1 \dots x_n$ $y_1 \dots y_n$ which differ from $x_1 \dots x_n$ in 1% of positions

Regular PCP may detect; Smooth will not see difference w.h.p.

Rectangular PCPs

Example: Max-2LIN.

Existence of Rectangular PCPs

Theorem (BHPT20)

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- *query complexity is $O(1)$*

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- *proof of length $2^n \operatorname{poly}(n)$*

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- *V is smooth*

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- *V is smooth*
- *V is almost rectangular*

Existence of Rectangular PCPs

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- *# of random bits is $n + o(n)$* — opt
- *query complexity is $O(1)$*
- *proof of length $2^n \operatorname{poly}(n)$*
- *V runs in time $2^{(1-\varepsilon)n}$*
- *V is smooth*
- *V is almost rectangular* — next time
- *the CSP problem is almost MAX-2LIN* $\equiv$ MAX-CUT