Matrix Rigidity

Rigidity of a Family of Circulant Matrices

Sasha Golovnev

November 18, 2020

Known Rigidity Bounds

For CLB, $R(\varepsilon n) \geq n^{1+\delta}$

- $\mathcal{R}(r) \geq \frac{n^2}{r} \log \frac{n}{r}$

Known Rigidity Bounds

- $\mathcal{R}(r) \geq \frac{n^2}{r} \log \frac{n}{r}$
- For linear $r = \varepsilon n$, $\mathcal{R}(\varepsilon n) \geq \Omega(n)$

Known Rigidity Bounds

- $\mathcal{R}(r) \geq \frac{n^2}{r} \log \frac{n}{r}$
- For linear $r = \varepsilon n$, $\mathcal{R}(\varepsilon n) \geq \Omega(n)$
- Most limitations say that we can't achieve $\mathcal{R}(\varepsilon n) \geq \boxed{n^{1+\delta}}$.

$\forall \delta \; \exists \alpha$ s.t.

$R_M(n^{1-\alpha}) < n^{1+\delta}$

$n^{1+o(1)}$ $\quad$ $n \log\log n$ $\quad$ $n \log^2 n$ $\quad$ $n \cdot 2^{\sqrt{\log n}}$

Known Rigidity Bounds

- $\mathcal{R}(r) \geq \frac{n^2}{r} \log \frac{n}{r}$
- For linear $r = \varepsilon n$, $\mathcal{R}(\varepsilon n) \geq \Omega(n)$
- Most limitations say that we can't achieve $\mathcal{R}(\varepsilon n) \geq n^{1+\delta}$
- What about just $\mathcal{R}(\varepsilon n) \geq \omega(n)$?

Known Rigidity Bounds

- $\mathcal{R}(r) \geq \frac{n^2}{r} \log \frac{n}{r}$
- For linear $r = \varepsilon n$, $\mathcal{R}(\varepsilon n) \geq \Omega(n)$
- Most limitations say that we can't achieve $\mathcal{R}(\varepsilon n) \geq n^{1+\delta}$

not nec $n^{1+\delta}$

- What about just $\mathcal{R}(\varepsilon n) \geq \omega(n)$?
- Would suffice for circuit lower bounds against series-parallel circuits

Family of Circulant Matrices

[CPR98] conjectured that an explicit family of circulant matrices has rigidity $\mathcal{R}(\varepsilon n) \geq \underline{n(\log n)^{\delta}}$.

Last week: No circulant matrix can have $R(\varepsilon n) \geq n^{1+\delta}$

$$\begin{pmatrix} \boxed{a_1 & a_2 & a_3 & a_4 & \dots & a_n} \\ a_2 & a_3 & a_4 & a_5 & \dots & a_1 \\ a_3 & a_4 & a_5 & a_6 & \dots & a_2 \\ a_4 & a_5 & a_6 & a_7 & \dots & a_3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_1 & a_2 & a_3 & \dots & a_{n-1} \end{pmatrix}$$

[CPR98] describe $a_1, \ldots, a_n \in \{0,1\}$

Only $O(\log n)$ of $a_1 \ldots a_n$ are ones, all the remaining entries are zeros.

In particular

$$\|M\|_0 = O(n \log n)$$

$$\Rightarrow R_M(r) \le O(n \log n) \quad \forall r$$

Thm $\boxed{\text{conj}}$ $R_M(n) \geq n (\log n)^{d}$

Odd Alternating Cycle Conjecture

Conjecture

[CPR98] For every field $\mathbb{F}$, there exists an odd k and $\varepsilon > 0$ such that every matrix $M \in \mathbb{F}^{n \times n}$ with non-zeros on the main diagonal and $\mathsf{rk}(M) \leq \varepsilon n$ contains an cycle of length k.

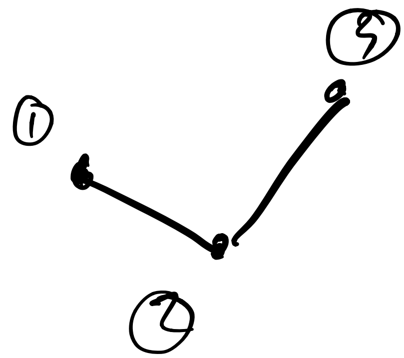
Odd Alternating Cycle Conjecture

Conjecture

[CPR98] For every field $\mathbb{F}$, there exists an odd k and $\varepsilon > 0$ such that every matrix $M \in \mathbb{F}^{n \times n}$ with non-zeros on the main diagonal and $\mathrm{rk}(M) \leq \varepsilon n$ contains an cycle of length k.

Rigidity over $\mathbb{F}$ over $\mathbb{F}$

[CPR98] This conjecture implies rigidity of an explicit circulant matrix



$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & 1 & \\ 2 & 1 & & 1 \\ 3 & & 1 & \end{array}$$

$$\begin{bmatrix} 1 & 1 & & & \\ 1 & 1 & & & \\ & & 1 & & 1 \\ & & & 1 & \\ & & 1 & & 1 \end{bmatrix}$$

If $rk(M) \leq \varepsilon n$, then G must contain a cycle of fixed odd length.

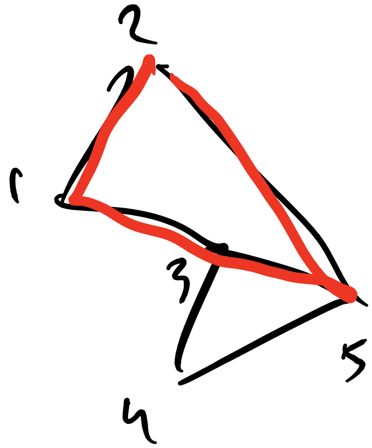
G without short cycles (of odd length) must have high rank

IF $rk(M)$ is $\leq \frac{n}{6} \Rightarrow$

G must contain a 4-cycle

For every even length of a cycle

G.



$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & & & \\ 2 & 1 & 1 & & & 1 \\ 3 & & & 1 & 1 & 1 \\ 4 & & & 1 & 1 & 1 \\ 5 & & 1 & 1 & 1 & 1 \end{array}$$

Conj. 1 If $rk(M) \leq \varepsilon n$
$\Rightarrow G$ must contain a triangle

[CPR98]: Counterexample: found graphs rk $n^{2/3}$ and without triangles.

[CPR98]. New Conjecture:
$\exists$ odd k, $\varepsilon > 0$ s.t.

If $rk(M) \leq \varepsilon n$
$\Rightarrow G$ must contain a cycle of length k

Counterexample to Conjecture

Theorem (GH20)

For every odd integer $k \geq 3$ there exists $\delta > 0$ such that for every sufficiently large integer n, there exists a matrix $M \in \mathbb{F}^{n \times n}$ with no odd cycle of length at most k such that for every finite field $\mathbb{F}$, $\mathbf{rk}_{\mathbb{F}}(M) \leq n^{1-\delta}$.

$\forall \mathbb{F}$ $\forall$ odd k we construct a graph with

(1) $rk(M) \leq n^{1-\delta} \Leftarrow$

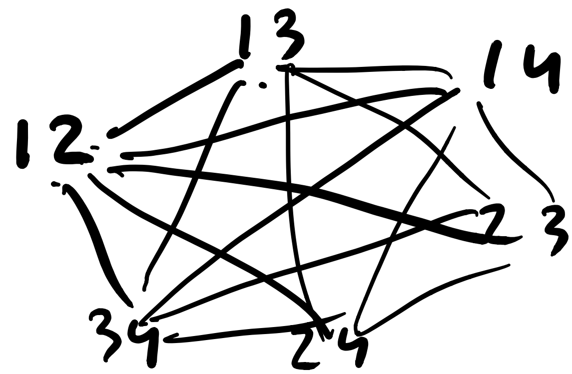
(2) no cycles of length k ...

Def Generalized Kneser graphs $K(d, \boxed{m})$.

Vertices are all subsets of $\{1, \dots, d\}$ of size $= \frac{d}{2}$.

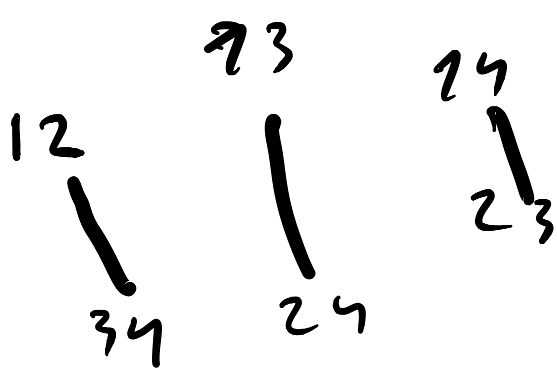
Two vertices $I, J \subseteq \{1, \dots, d\}$ are connected iff $|I \cap J| < m$.

Ex $d = 4$ $m = 2$.



$\cong K_6$

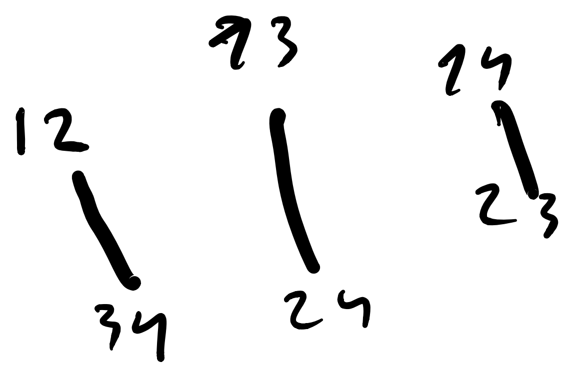
Ex $d = 4$ $m = 1$



– Matching 6

Let's consider $K(d,m)$

Let M be its adjacency matrix with 1s on the main diagonal.



— $K(4,1)$

$K(d,m)$

$$M = \begin{array}{c|cccccc} & 12 & 13 & 14 & 23 & 24 & 34 \\ \hline 12 & 1 & & & & & 1 \\ 13 & & 1 & & & 1 & \\ 14 & & & 1 & 1 & & \\ 23 & & & 1 & 1 & & \\ 24 & & 1 & & & 1 & \\ 34 & 1 & & & & & 1 \end{array}$$

$$\mathrm{Rk}(M) \le \sum_{i=0}^{\frac{d}{2}-m} \binom{d}{i}$$

$K(d, m)$

$$\# \text{vertices} = \binom{d}{d/2} \approx \frac{2^d}{\sqrt{d}} = 2^{d - o(d)}$$

$m = \varepsilon d$

$$Rk(M) = \sum_{i=0}^{d(\frac{1}{2}-\varepsilon)} \binom{d}{i} \leq$$

$$\leq 2^{d \underline{H(\frac{1}{2}-\varepsilon)}} \approx 2^{d(1-\varepsilon^2)}$$

$$= \boxed{2^{d(1-\delta)}}$$

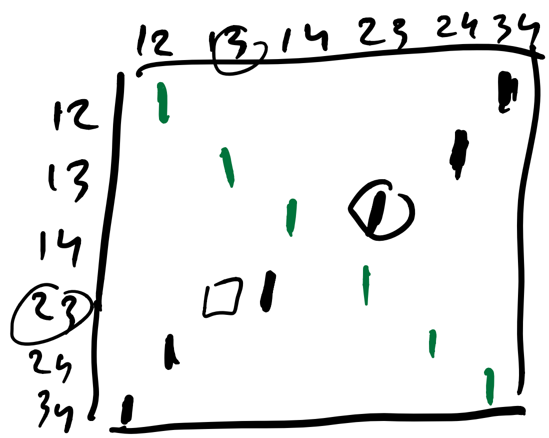
$\#$ vertices is $N \approx 2^d$

rk is $\leq N^{(1-\delta)}$

$m = \varepsilon d << \frac{d}{2}$

$|I| = |J| = d/2$

$I, J \subseteq \{1..d\}$

$M =$ 

$M_{I,J} =$

$= f(I,J) = \boxed{\prod_{i=m}^{d/2-1} \left(|I \cap J| - i\right)}$

Proof: (1) Green ones – on the main diag.

$M_{I,I} = f(I,I) =$

$= \prod_{i=m}^{d/2-1} \left(d/2 - i\right) \neq 0$

(2) $M_{I,J} = f(I,J) = 0$

IFF $|I \cap J| \geq m$

$$= \boxed{\prod_{i=m}^{d/2-1} \left(|I \cap J| - i \right)}$$

$$\underline{m \leq} |I \cap J| \underline{\leq \frac{d}{2} - 1}$$

$$|I| = |J| = \frac{d}{2} \Rightarrow |I \cap J| \underline{\leq \frac{d}{2} - 1}$$

$$M_{I,J} = 0 \quad \text{IFF} \quad |I \cap J| \geq m$$

$$M_{I,J} \neq 0 \quad \text{IFF} \quad |I \cap J| < m$$

def of an edge in the Kneser graph $K(d, m)$

$P_{x,y} = p(x,y)$ where p has m monomials $\Rightarrow$

$\mathrm{rk}(P) \leq m$

Our deg $\left(\frac{d}{2}-1\right) - m + 1 \approx \frac{d}{2} - m$

monomials

$$\boxed{\sum_{i=0}^{\frac{d}{2}-m} \binom{d}{i}}$$

$\Rightarrow$

$$\mathrm{Rank}(M) \leq \sum_{i=0}^{\frac{d}{2}-m} \binom{d}{i}$$

$$= 2^{dH\left(\frac{1}{2}-\varepsilon\right)} = 2^{d(1-\delta)}$$

sublinear in # of vertices

k - Fixed odd constant

$m = \frac{d}{2k}$, consider

$K(d, m)$

Claim has no cycles of odd length $\leq k$.

$k = 3$

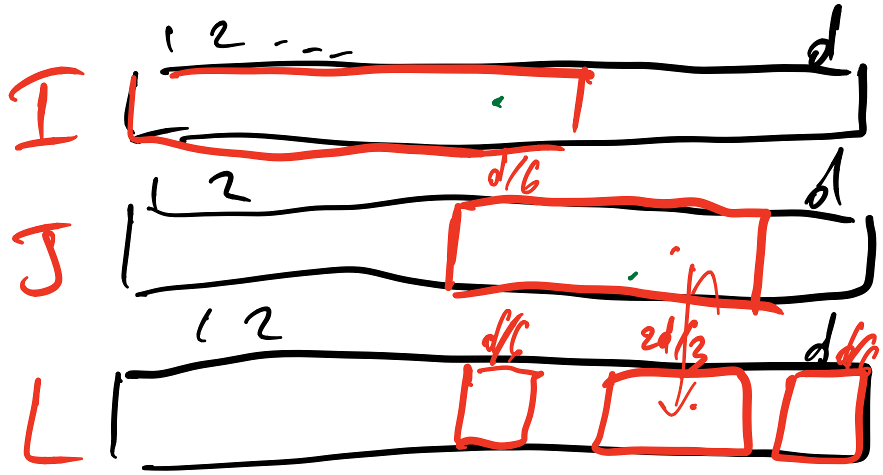
$K(d, \frac{d}{6})$

$|I \cap J| < \frac{d}{6}$

Assume $I - J - L$

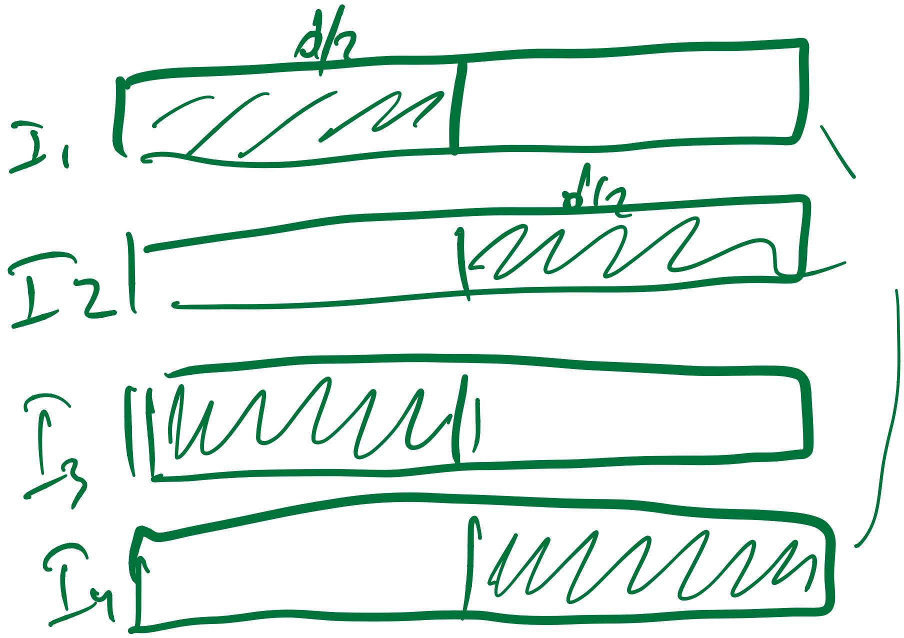
$|I| = |J| = |L| = \frac{d}{2}$

$|I \cap J|, |I \cap L|, |J \cap L| < \frac{d}{6}$



Even length

$$\underbrace{I_1 - I_2 - I_3 - I_4 - I_5 - I_6}$$



If k is any odd number

then

$|I_1 \cap I_k|$ is large — they are not connected $\Rightarrow$ no cycles of length k.